A cyclotron resonance mechanism for very-low-frequency whistler-mode sideband wave radiation. III. Effect of the inhomogeneity on sideband resonances

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This part of the paper addresses the effects of the inhomogeneity of the magnetic field of the earth on the resonances studied previously. We find that the inhomogeneity introduces phase shifts on the equilibrium positions of both types of resonance we have treated. Those phase shifts are time dependent if either the main carrier in the case of internal resonances, or the interacting carriers in the case of external resonances have time-dependent amplitudes. The time-dependent phase shifts will translate themselves into sideband wave frequency shifts. If the carriers have constant amplitudes, the phase shifts are locally constant. However, frequency shifts observable at the end of the interaction region can still be produced if the amplitude of the radiated sidebands are locally time dependent. Since the phase shifts are dependent in $v_\perp$, the sideband waves should undergo an increase in width as they shift away from their nominal positions. A strong enough inhomogeneity will destroy a resonance completely. It is found that trapped resonances are much more resilient to the effects of the inhomogeneity than external resonances, this being one of the main reasons why we can experimentally observe the creation of sizable sidebands by extremely weak carriers.

I. INTRODUCTION

This part of the paper will analyze the effect of the inhomogeneity of the magnetic field of the earth on the resonances studied in parts I and II. The analytical and numerical methods used are the same as in part I, and so is the notation. A familiarity with parts I and II of the paper is required.

In Sec. II of this third and last part, phase plots are used to show graphically the effect of the inhomogeneity on the resonances we had previously studied in a homogeneous medium. It is seen that those effects consist mainly of phase shifts superposed on the resonance positions, coupled with complete resonance destruction if the value of the inhomogeneity is large enough. In Sec. III the electron equations of motion are rewritten to include the inhomogeneity, and in Sec. IV the Lie perturbation method is applied to them to study analytically the effect of the inhomogeneity on the "trapped" resonances. In Sec. V we explain how the phase shifts created by the inhomogeneity can, even when locally constant in time, produce frequency shifts and line broadening in the sideband radiation received at the end of the interaction region. In Sec. VI we apply the Lie perturbation method to external resonances obtaining results similar to the ones found in Secs. IV and V for internal resonances. Section VII takes a look at some general properties of higher-order resonances, and Sec. VIII contains the conclusions, a chief one among them being that the inhomogeneity is a principal cause for the peculiar but experimentally observed fact that vanishingly small carriers can create sizable sidebands if the sideband producing interaction includes at least one strong carrier.

II. PHASE PLOTS AND RELATION BETWEEN FREQUENCY AND PHASE SHIFTS

Figure 1 shows phase plots for an $l = 2$ first-order "trapped" resonance created by perturbing the main wave with a weaker carried located at $\dot{\theta} = \Omega$. This resonance will radiate mainly at the frequencies $\pm\Omega$ and $\pm\Omega/2$. Figure 1(a) shows the phase plot for the zero inhomogeneity case, where we can see the two-lobed structure of the resonance inside the main wave potential well, and the perturbing carrier to the right, shifted 180° in phase relative to the main wave.

Figure 1(b) shows the effect of a positive inhomogeneity (a positive gradient of the intensity of the magnetic field of the Earth) on the resonances. The resonance associated with the perturbing carrier has been completely wiped out, and therefore such a carrier will experience no growth. However, the "trapped" resonance that it creates is still present and can give rise to growing lines. Since the original amplitude of the perturbing carrier is only 2% (-33 dB) of the main carrier, the weak carrier will not be easily seen in the sideband spectrum, but the sidebands it creates will. In the same picture it can be seen that the outermost phase trajectories have been stripped away from the main carrier resonance and that the two lobes in the "trapped" resonance have acquired different negative phase shifts. The first effect is widely known, the second one will be described in detail in Sec. IV. There is also a small inward shift of the trapped resonance and a positive phase shift (upward displacement) of the main carrier resonance. This phase shift can easily be shown to be equal to $\sin^{-1}(\tau/\Omega_{t1}^2)$ where τ is the torque associated with the inhomogeneity (see Sec. IV), and Ω_{t1} is the main wave trapping frequency. The internal resonance phase shift will affect the frequency separation between radiated harmonics and subharmonics. The phase shift in the main resonance will affect the main carrier frequency. (The mechanism for translation of phase shifts into frequency shifts is discussed in Sec. V.)

Figure 1(c) shows the consequence of moving the perturbing carrier form Ω to $-\Omega$. The equilibrium points in

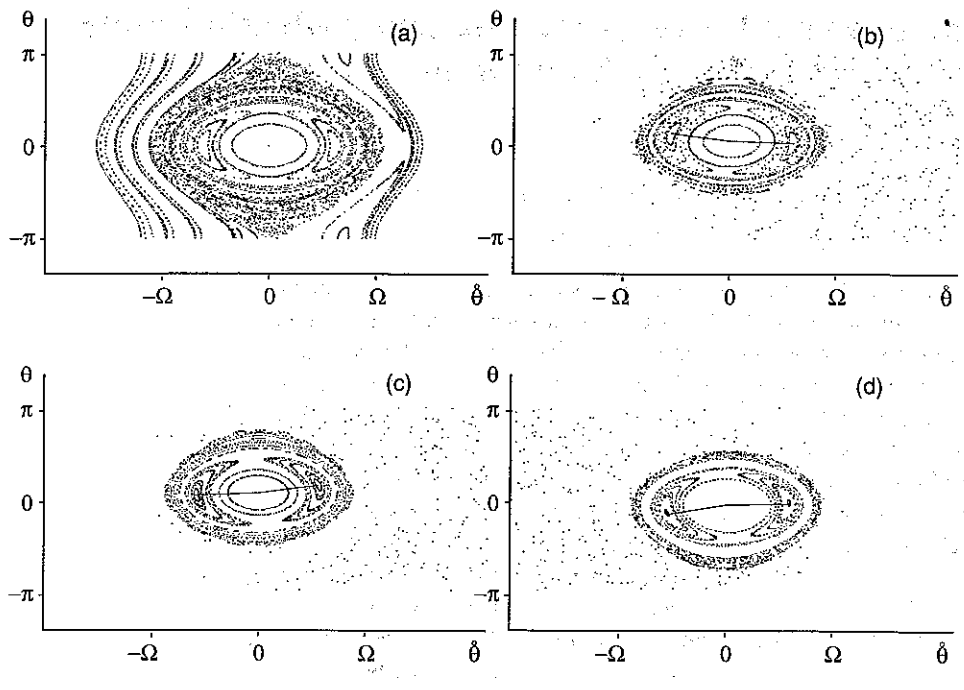

FIG. 1. Effect of the inhomogeneity on a half harmonic "trapped" resonance. The main carrier is located at $\dot{\theta} = 0$. The main carrier trapping frequency is slightly above $\Omega/2$, and the weak to main carrier amplitude ratio is 0.02(− 34 dB). The weak carrier is shifted in phase by 180° relative to the main carrier: (a) $\tau/\Omega_t^2 = 0$. The weaker carrier is located at $\dot{\theta} = \Omega$. Chaos, the perturbing wave resonance, and the two-lobed half harmonic can be readily seen. (b) $\tau/\Omega_t^2 = 0.08$. The weaker carrier is still located at $\dot{\theta} = \Omega$. Chaos and the perturbing wave resonance have been swept away, but the resonance that the latter creates is still present. Both resonance lobes have acquired a negative phase shift. The main wave resonance has a slight shift upwards. (c) $\tau/\Omega_t^2 = 0.08$. The weaker carrier is now located at $\dot{\theta} = -\Omega$. The resulting plot can be obtained from (b) by a reflection around a vertical axis going through $\dot{\theta} = 0$. As a consequence, the two lobes have their shifts swapped and changed in sign. (d) $\tau/\Omega_t^2 = -0.08$. The weaker carrier is at $\dot{\theta} = \Omega$. The plot can be obtained by reflecting (b) around a horizontal axis going through $\theta = 0$. Both lobes and the main wave resonance have their phase shifts changed in sign.

the resonance are reflected around a vertical axis going through $\dot{\theta} = 0$. As a consequence, both their phase shifts become positive.

Figure 1(d) shows the effect on the phase plot of changing the sign of the inhomogeneity, but keeping the perturbing wave at the frequency Ω. The resonance equilibrium points are reflected around a horizontal axis going through $\theta = 0$. That implies a change in sign of each phase shift in the "trapped" resonance and a change in sign of the main wave upward shift. The total phase shift associated with each radiated sideband is consequently multiplied by minus one.

Figure 2 compares the effects of the inhomogeneity on "trapped" and external resonances having the same approximate initial widths. Figure 2(a) shows a three-wave resonance, already described in part II for the case of zero inhomogeneity. Figure 2(b) shows the effect on it of a fairly large inhomogeneity. As in Fig. 1, it can be seen that the external resonances are completely wiped out, that there is a phase shift, now positive, added to each resonance lobe, that external phase trajectories are stripped away, and that, now in a more pronounced way, there is an upward shift of the main wave resonance together with an appreciable inward motion of the internal resonance. This inward displacement protects and extends the lifetime of the resonance by moving it away from the outward trajectories that are soon destroyed by the inhomogeneity. The inhomogeneity in this case is $0.18\,\Omega_{t1}^2$.

Figure 2(c) shows several external resonances between two equal intensity carriers for the case of zero inhomogeneity. Figure 2(d) shows the effect of an inhomogeneity just strong enough to destroy all external resonances including the half harmonic. The value of the inhomogeneity in this case is only $0.03\,\Omega_{t1}^2$. Since the resonance shown in Fig. 2(a) is in reality still present for inhomogeneities of the order of $0.25\,\Omega_{t1}^2$, and since the size of the interaction region is proportional to the maximum inhomogeneity that a resonance can withstand, we arrive at the important conclusion that "trapped" resonances can exist over an interaction region almost 10 times as large as external resonances.

III. MODIFICATION OF THE ELECTRON EQUATION OF MOTION BY A NONZERO INHOMOGENEITY

The effect of the inhomogeneity on the electron motion can be locally described by a time-dependent torque[1–4] applied to the perturbed equations developed in the preceding parts of this paper. The equation describing the electron motion under the influence of N waves will be then

$$\ddot{\theta} = \tau(t) - \sum_{i=1}^{N} \Omega_{ti}^2 \sin(\theta - \Omega_i t + \phi_i), \qquad (1)$$

where τ is the inhomogeneity.

For the study of external resonances, the $(\theta,\dot{\theta}) = (q,p)$ pair of coordinates is maintained and an equivalent Hamiltonian can be written:

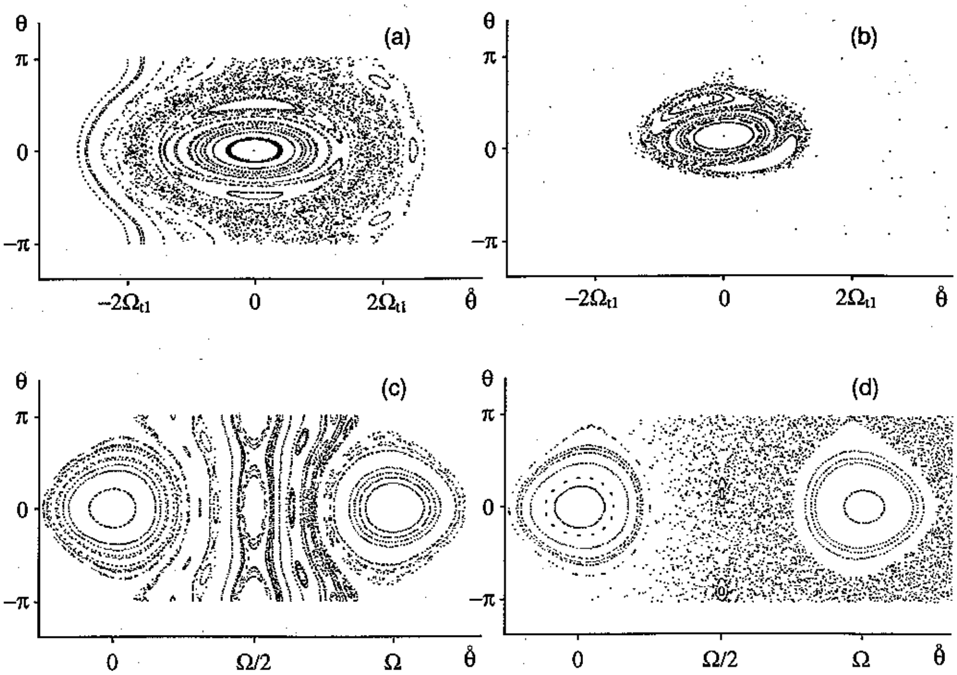

FIG. 2. Relative effects of the inhomogeneity on "trapped" and external resonances: (a) $\tau/\Omega_t^2 = 0.0$. Second-order two-lobed resonance created by the interaction of three different carriers. (b) $\tau/\Omega_t^2 = 0.18$. Despite the existence of a fairly large inhomogeneity, the resonance is still intact. The most apparent effects on it are the creation of a positive phase shift, and a displacement of its orbit towards the center of the main wave. An upward shift of the main wave resonance can also be seen. (c) $\tau/\Omega_t^2 = 0.0$. Half and one-third harmonic resonances created outside the potential wells of two equal amplitude waves. (d) $\tau/\Omega_t^2 = 0.03$. An extremely small inhomogeneity completely wipes out the one-third and almost completely destroys the half harmonic resonance.

$$h(q,p,t) = \frac{p^2}{2} - \tau q - \sum_{i=1}^{N} \Omega_{ti}^2 \cos(q - \Omega_i t + \phi_i). \qquad (2)$$

For the study of internal resonances created by the wave-wave interaction process, we will use the (ϕ, j) pair of canonical coordinates that gives rise to the following Hamiltonian:

$$h(\phi,j,t) = h_0(j) - \tau\theta(\phi,j) - \sum_{i=2}^{N} \sum_{n=-\infty}^{\infty} \Omega_{ti}^2 V_n(j) \times \cos(n\phi - \Omega_i t + \phi_i), \qquad (3)$$

where

$$\theta = 2\sin^{-1}\left[\kappa\, \mathrm{sn}\left(\frac{2K(\kappa)\phi}{\pi}\right)\right] \qquad (4)$$

can be rewritten as[5]

$$\theta = \sum_{n=1}^{\infty} b_n(j)\sin[(2n-1)\phi], \qquad (5)$$

with

$$b_n(j) = 2\left(\frac{\Omega_{t1}}{\Omega_t}\right)^2 \frac{V_{2n-1}(j)}{(2n-1)^2}. \qquad (6)$$

Those equations will be used to study analytically the inhomogeneity effects observed in the phase plots described in the preceding sections.

IV. FIRST-ORDER INTERNAL RESONANCES

If we put $\Omega_{ti}^2 = \epsilon A_i$, we can write the Hamiltonian as

$$h(\phi,j,t) = h_0(j) + \epsilon h_1(\phi,j,t), \qquad (7)$$

with

$$h_1 = -\tau\theta(\phi,j) - \sum_{i=2}^{N} \sum_{n=-\infty}^{\infty} A_i V_n(j)\cos(n\phi - \Omega_i t + \phi_i). \qquad (8)$$

[We have substituted $\epsilon\tau$ for τ in $h(\phi,j,t)$ to explicitly show that perturbations from the inhomogeneity and from the additional carriers on the main wave have the same order of magnitude.]

To study first-order resonances we put

$$K_0 = h_0 \qquad (9)$$

and choose

$$K_1 = -A_j V_l(J)\cos(l\Phi - \Omega_j t + \phi_j). \qquad (10)$$

This implies

$$W_1 = -\int^t d\tau\{h_1[\Phi + \Omega_t(\tau - t), J, \tau] - K_1[\Phi + \Omega_t(\tau - t), J, \tau]\}$$

or

$$W_1 = -\frac{\tau}{\Omega_t(J)} \sum_{n=1}^{\infty} \frac{b_n(J)}{(2n-1)}\cos[(2n-1)\Phi] + \sum_{\substack{(i\neq j)\text{or}\\(n\neq l)}} a_{in}\sin(n\Phi - \Omega_i t + \phi_i), \qquad (11)$$

with

$$a_{in} = \frac{A_i V_n(J)}{n\Omega_t(J) - \Omega_i}. \qquad (12)$$

The corrections to the resonance due to the inhomogeneity are second-order terms that must be transferred to K_2 from

$$\{W_1, h_1 + K_1\} = \frac{\partial W_1}{\partial \Phi}\frac{\partial(h_1 + K_1)}{\partial J} - \frac{\partial W_1}{\partial J}\frac{\partial(h_1 + K_1)}{\partial \Phi}, \qquad (13)$$

where $h_1 + K_1$ can be written as

$$h_1 + K_1 = -\tau\theta(\Phi,J) - \sum_{i,n}(1 + \delta_{ij}\delta_{nl})A_i V_n \times \cos(n\Phi - \Omega_i t + \phi_i). \qquad (14)$$

Dfferentiating the above expression with respect to Φ and J, inserting the results in Eq. (13), transforming products of trigonometric functions into sums and differences, and putting into K_2 all resulting time independent and resonant terms, we get

$$2K_2 = 2K_2^0 - \frac{\tau^2}{2\Omega_t}\frac{d}{dJ}\sum_{n=1}^{\infty} b_n^2 + \frac{\tau A_j}{2\Omega_t} \times \sum_{m=1}^{\infty}\left(\frac{[b'_m - b_m(\Omega'_t/\Omega_t)]}{(2m-1)}(n_+V_{n_+} + n_-V_{n_-}) + b_m(V'_{n_+} - V'_{n_-})\right)\sin(l\Phi - \Omega_j t + \phi_j), \tag{15}$$

where $n_{\pm} = l \pm (2m-1)$. Computer analysis shows that keeping only the first term in both summations leads to a very good approximation to the equation above. If we take into consideration that

$$b'_1 = b_1\left(\frac{V'_1}{V_1} - 2\frac{\Omega'_t}{\Omega_t}\right), \tag{16}$$

we get for K_2:

$$K_2 = K_2^0 - \frac{\tau^2}{4\Omega_t}\frac{db_1^2}{dJ} + \frac{\tau}{2}\frac{b_1 A_j}{\Omega_t} \times \left[\left(\frac{V'_1}{V_1} - \frac{5}{2}\frac{\Omega'_t}{\Omega_t}\right)[(l+1)V_{l+1} + (l-1)V_{l-1}] + (V'_{l+1} - V'_{l-1})\right]\sin(l\Phi - \Omega_j t + \phi_j). \tag{17}$$

It will be convenient to transform the derivatives in the above expression into derivatives with respect to the adimensional variable κ^2, using the operator identity

$$\frac{\partial}{\partial J} = \frac{\Omega_t}{2\Omega_{t1}^2}\frac{\partial}{\partial \kappa^2}. \tag{18}$$

We can then write

$$K_2 = K_2^0 - \frac{1}{8}\left(\frac{\tau}{\Omega_{t1}}\right)^2\frac{db_1^2}{d\kappa^2} + \tau\frac{A_j}{\Omega_{t1}^2}f_l(\kappa^2) \times \sin(l\Phi - \Omega_j t + \phi_j), \tag{19}$$

with

$$f_l(\kappa^2) = \frac{b_1}{4}\left[\left(\frac{V'_1}{V_1} - \frac{5}{2}\frac{\Omega'_t}{\Omega_t}\right)[(l+1)V_{l+1} + (l-1)V_{l-1}] + (V'_{l+1} - V'_{l-1})\right], \tag{20}$$

where a primed variable, in the above equation only, means it is differentiated with respect to κ^2. The total Hamiltonian will be

$$K = h_0 - \epsilon A_j V_l\cos(l\Phi - \Omega_j t + \phi_j) + \epsilon^2 K_2^0 - \frac{\epsilon^2}{8}\left(\frac{\tau}{\Omega_{t1}}\right)^2\frac{db_1^2}{d\kappa^2} + \epsilon^2\tau\frac{A_j}{\Omega_{t1}^2}f_l(\kappa^2) \times \sin(l\Phi - \Omega_j t + \phi_j). \tag{21}$$

If we define

$$\Phi_0(l,\kappa^2) = \frac{\epsilon\tau}{\Omega_{t1}^2}\frac{f_l(\kappa^2)}{lV_l(\kappa^2)} = \frac{\epsilon\tau}{\Omega_{t1}^2}\Phi_{\text{norm}} \tag{22}$$

and assume $\sin(l\Phi_0) \approx l\Phi_0$, we can finally write

$$K = h_0 + \epsilon^2 K_2^0 - \frac{\epsilon^2}{8}\left(\frac{\tau}{\Omega_{t1}}\right)^2\frac{db_1^2}{d\kappa^2} - \epsilon A_j V_l \times \cos[l(\Phi + \Phi_0) - \Omega_j t + \phi_j]. \tag{23}$$

Figure 3 shows plots of Φ_{norm} as a function of κ^2 for a few low values of l and also the asymptotic value when $l \to \infty$.

From the way Φ_0 was calculated, it is possible to have an understanding of the meaning of such a phase shift. Because the inhomogeneity is only a perturbation, the electron stream motion is mainly controlled by the main wave. The oscillatory motion imparted on the stream inside the main wave potential well makes the potential from the inhomogeneity, which is linear in θ, seem oscillatory with a frequency $\dot{\Phi}$. The first term in the series for this effective oscillatory potential is equivalent to a first-order resonance with $l = 1$ and $\Omega_i = 0$. The phase shift we are looking for is a second-order effect that comes from the interaction of this effective resonance with two first-order resonances having parameters $(l+1,\Omega_j)$ and $(l-1,\Omega_j)$. Their interaction gives rise to a second-order resonance with parameters (l,Ω_j), shifted by $-90°$ in phase relative to the initial trapped resonance. The coherent superposition of these two resonances gives rise to the phase-shifted resonance.

Hamilton's equations applied to K will give

$$\dot{J} = -\frac{\partial K}{\partial \Phi} = -\epsilon l A_j V_l(J)\sin[l(\Phi + \Phi_0) - \Omega_j t + \phi_j], \tag{24}$$

$$\dot{\Phi} = \frac{\partial K}{\partial J} = \frac{dh_0(J)}{dJ} + \epsilon^2\frac{dK_2^0}{dJ} - \frac{\epsilon^2}{16}\left(\frac{\tau}{\Omega_{t1}^2}\right)^2\Omega_t\frac{d^2 b_1^2}{d(\kappa^2)^2} - \epsilon A_j\frac{dV_l(J)}{dJ}\cos[l(\Phi + \Phi_0) - \Omega_j t + \phi_j] + \epsilon l A_j V_l(J)\frac{d\Phi_0}{dJ}\sin[l(\Phi + \Phi_0) - \Omega_j t + \phi_j]. \tag{25}$$

Since

$$\ddot{\Phi} = \frac{\partial\dot{\Phi}}{\partial J}\dot{J} + \frac{\partial\dot{\Phi}}{\partial\Phi}\dot{\Phi} + \frac{\partial\dot{\Phi}}{\partial t}, \tag{26}$$

we can put

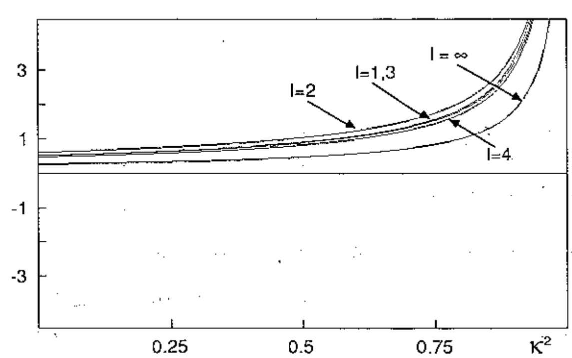

FIG. 3. Plot of $\Phi_{\text{norm}} = (\Omega_{t1}^2/\tau)\Phi_0$, the normalized constant part of the phase shift caused by the action of the inhomogeneity on first-order resonances. The picture shows plots for $l = 1...4$ and the asymptotic value $l = \infty$. The divergences at $\kappa^2 = 1$ indicate the breakdown of the expressions near the separatrix due to the existence of chaos and the stripping away of closed phase orbits by the inhomogeneity.

$$\ddot{\Phi} \approx \frac{d^2 h_0(J)}{dJ^2} \dot{J}$$

$$= -\epsilon \frac{d^2 h_0(J)}{dJ^2} l A_j V_l(J) \sin[l(\Phi+\Phi_0) - \Omega_j t + \phi_j] . \quad (27)$$

The equation describes an $|l|$-lobed resonance oscillating inside the main wave potential well with frequency $\Delta\omega_j/l$ and shifted in phase by $-\Phi_0$. ($\Delta\omega_j$ is the frequency separation between the perturbing wave j and the main wave.)

From the $\dot{\Phi}$ equation we also get at resonance

$$\frac{\Omega_j}{l} = \Omega_t + \epsilon A_j \frac{dV_l}{dJ} + \epsilon^2 \frac{dK_2^0}{dJ} - \frac{\epsilon^2}{16} \left(\frac{\tau}{\Omega_{t1}^2}\right)^2 \Omega_t \frac{d^2 b_1^2}{d(\kappa^2)^2}. \quad (28)$$

Defining J_0 as the value of J associated with the resonance as $\epsilon \to 0$, and expanding Ω_t around this value, we get

$$\Omega_t(J) = \Omega_t(J_0) - \left|\frac{d\Omega_t}{dJ}\right| \Delta J = \frac{\Omega_j}{l} - \left|\frac{d\Omega_t}{dJ}\right| \Delta J. \quad (29)$$

Substituting the last expression into Eq. (28), we can calculate J and ΔJ:

$$J = J_0 + \Delta J = J_0 + \Delta J_{\text{hom}} + \Delta J_{\text{inhom}}, \quad (30)$$

where

$$\Delta J_{\text{hom}} = \left(\epsilon A_j \frac{dV_l}{dJ} + \epsilon^2 \frac{dK_2^0}{dJ}\right) \left|\frac{d\Omega_t}{dJ}\right|^{-1} \quad (31)$$

and

$$\Delta J_{\text{inhom}} = -\frac{\epsilon^2}{16} \Omega_t \left|\frac{d\Omega_t}{dJ}\right|^{-1} \left(\frac{\tau}{\Omega_{t1}^2}\right)^2 \frac{d^2 b_1^2}{d(\kappa^2)^2}. \quad (32)$$

Since the second derivative of b_1^2 is positive, in the (Φ, J) coordinate system the inhomogeneity has the effect of decreasing the value of J, i.e., of pulling the resonance towards the center of oscillation.

To look at the stream motion in the (ϕ, j) variables, we need the transformation equations correct to second order:

$$\phi = \Phi - \epsilon \frac{\partial W_1}{\partial J} - \frac{\epsilon^2}{2} \left\langle \left\{ W_1, \frac{\partial W_1}{\partial J} \right\} \right\rangle, \quad (33)$$

$$j = J + \epsilon \frac{\partial W_1}{\partial \Phi} + \frac{\epsilon^2}{2} \left\langle \left\{ W_1, \frac{\partial W_1}{\partial \Phi} \right\} \right\rangle, \quad (34)$$

where the angle brackets mean as always that the time-independent part of the expression should be taken.

Evaluating specific terms we obtain

$$\frac{\partial W_1}{\partial \Phi} = \frac{\tau}{\Omega_t} \sum_{n=1}^{\infty} b_n \sin[(2n-1)\Phi] + \sum_{\substack{(i \neq j) \text{or} \\ (n \neq l)}} n a_{in} \cos(n\Phi - \Omega_i t + \phi_i), \quad (35)$$

$$\frac{\partial W_1}{\partial J} = -\tau \sum_{n=1}^{\infty} \left(\frac{b_n}{\Omega_t}\right)' \frac{1}{(2n-1)} \cos[(2n-1)\Phi] + \sum_{\substack{(i \neq j) \text{or} \\ (n \neq l)}} a'_{in} \sin(n\Phi - \Omega_i t + \phi_i), \quad (36)$$

$$\left\langle \left\{ W_1, \frac{\partial W_1}{\partial \Phi} \right\} \right\rangle = \frac{\tau^2}{\Omega_t} \sum_{n=1}^{\infty} b_n \left(\frac{b_n}{\Omega_t}\right)' + \sum_{\substack{(i \neq j) \text{or} \\ (n \neq l)}} n^2 a_{in} a'_{in}, \quad (37)$$

$$\left\langle \left\{ W_1, \frac{\partial W_1}{\partial J} \right\} \right\rangle = 0. \quad (38)$$

For the $J \to j$ transformation, we drop the first-order time-dependent terms, we keep only the first term of the time-independent series in the expression for $\partial W_1/\partial\Phi$, and transform all derivatives into derivatives with respect to κ^2. We then get

$$j = j_{\text{hom}} + \epsilon \frac{\tau}{\Omega_t} b_1 \sin\Phi - \frac{\epsilon^2}{2} \Omega_t \left(\frac{\tau}{\Omega_{t1}^2}\right)^2 \frac{d^2 b_1^2}{d(\kappa^2)^2} + \frac{\epsilon^2}{4} \Omega_{t1} \left(\frac{\tau}{\Omega_{t1}^2}\right)^2 b_1 \frac{d}{d\kappa^2} \left(\frac{\Omega_{t1} b_1}{\Omega_t}\right), \quad (39)$$

where

$$j_{\text{hom}} = J_0 + \Delta J_{\text{hom}} + \sum_{\substack{(i \neq j) \text{or} \\ (n \neq l)}} n^2 a_{in} \frac{da_{in}}{dJ}, \quad (40)$$

which can be rewritten as

$$j = j_{\text{hom}} + \epsilon \frac{\tau}{\Omega_t} b_1 \sin\Phi + \epsilon^2 \Omega_{t1} \left(\frac{\tau}{\Omega_{t1}^2}\right)^2 \Delta j_{\text{norm}}. \quad (41)$$

If τ is positive, the oscillatory term linear in ϵ is positive when θ is positive and negative otherwise. Its effect is to push all points in the phase trajectories upwards, and essentially describes the phase shift of the main wave resonance observed in the phase plots of Figs. 1 and 2. The constant second-order term describes a systematic shift of the "trapped" resonance towards a different orbit. Figure 4 has a plot of Δj_{norm}, which is adimensional and depends on κ^2 only. The plot shows it to be negative and therefore describing a displacement of the resonance towards the center of oscillation.

For the $\Phi \to \phi$ transformation we can write $\Phi = \Phi_{\text{hom}} - \Phi_0$, where Φ_{hom} is the value of Φ obtained from Hamilton's equation when $\tau = 0$. We then throw away all explicitly time-dependent terms and keep only the first term in the remaining series expansion for $\partial W_1/\partial J$ to get

$$\phi = \Phi_{\text{hom}} - \Phi_0 + \epsilon\tau (b_1/\Omega_t)' \cos\Phi. \quad (42)$$

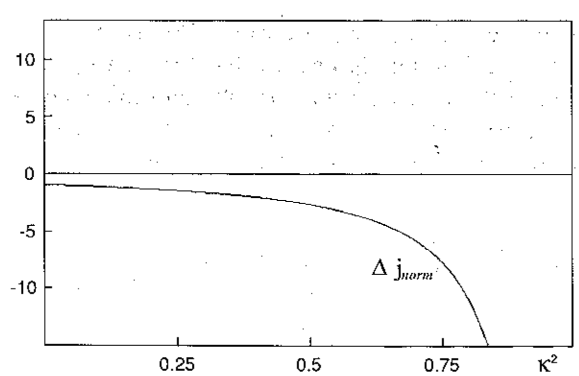

FIG. 4. Plot of Δj_{norm}, the normalized constant shift in j due to the action of the inhomogeneity on the main wave potential. The shift is negative and independent of the specific resonance under consideration. The divergence at $\kappa^2 = 1$ is due to chaos and the presence of the inhomogeneity that combined destroy the outer electron orbits of the main resonance rendering the analytical expressions inadequate.

We see that added to the constant phase shift $-\Phi_0$, there is a periodic term that for positive τ is positive when θ is positive and negative otherwise. For the two-lobed resonances shown in Figs. 1 and 2, this term advances the motion of an equilibrium point at the right of the axis $\dot{\theta} = 0$ and retards its motion at the left. This term is the cause of the different phase shifts observed for the two lobes of the resonance in Fig. 1. Such a term can be understood as a consequence of the upward shift of the main wave resonance that decreases the j values of the lower part of the phase trajectory, increasing the values of the trapping frequency, and does the opposite on top.

The expression above explains the values and reflection properties of the phase shifts observed in the phase plots of Fig. 1:

We can rewrite it as

$$\phi = \Phi_{\text{hom}} - \Phi_0 + \tau a \cos \Phi = \Phi_{\text{hom}} + \Delta\phi. \tag{43}$$

Φ_{hom} is the equilibrium point of the resonances for $\tau = 0$. For the resonance located at $\Phi_{\text{hom}} = 0$, the phase shift will be

$$\Delta\phi_1 = -\Phi_0 + \tau a \cos \Phi_0, \tag{44}$$

and for the resonance at $\Phi_{\text{hom}} = \pi$,

$$\Delta\phi_2 = -\Phi_0 - \tau a \cos \Phi_0. \tag{45}$$

We see that in $\Delta\phi_1$ the two contributing terms have opposite signs and tend to cancel. For $\Delta\phi_2$ they add. This makes $|\Delta\phi_2| > |\Delta\phi_1|$ as seen in Fig. 1(b).

When the perturbing wave is shifted from Ω to $-\Omega$, l changes to $-l$. It is fairly easy to show that Φ_0 is odd under such a transformation, i.e., $\Phi_0(-l,\kappa^2) = -\Phi_0(l,\kappa^2)$. The effects on the phase shifts are

$$\Delta\phi_1 \rightarrow \Phi_0 + \tau a \cos \Phi_0 = -\Delta\phi_2 \tag{46}$$

and

$$\Delta\phi_2 \rightarrow \Phi_0 - \tau a \cos \Phi_0 = -\Delta\phi_1. \tag{47}$$

Those two equations define the reflection transformation seen in Fig. 1(c).

When the inhomogeneity is multiplied by -1, both terms change sign because they are proportional to τ. This implies

$$\Delta\phi_{1,2} \rightarrow -\Delta\phi_{1,2} \tag{48}$$

that are the reflection properties seen in Fig. 1(d).

As will be shown in the next section, the relevant part of the phase shifts is the constant part, Φ_0, and the important result to be recalled is that Φ_0 changes sign under either of the transformations considered above and represented in the phase plots of Fig. 1.

V. INTERNAL RESONANCE RADIATION FREQUENCY SHIFTS

The internal resonance phase shifts are a sum of two terms: a constant part, Φ_0, common to all points in the resonance, and an oscillatory part that depends on the angular location of each lobe. The oscillatory part is the cause of the unequal phase shifts in the phase plots described in Sec. II. Its presence will modulate the resonance motion producing additional radiation falling on already established sideband positions. This radiation will not affect the profile of the radiated spectrum to a great extent and its presence will be neglected. We will concentrate instead on the effect that Φ_0 has on the resonance radiation.

If Ω_{t1} is time dependent, Φ_0 will also be time dependent, giving origin to frequency shifts. Reminding ourselves that Ω_t remains equal to Ω_j/l, independently of the variations in the wave trapping frequency, we can write

$$\Omega_{t1} = \frac{2\Omega_j K(\kappa)}{\pi l} \tag{49}$$

and

$$\dot{\Omega}_{t1} = \frac{\Omega_{t1}}{K(\kappa)} \frac{dK(\kappa)}{d\kappa^2} \frac{d\kappa^2}{dt}. \tag{50}$$

Differentiating Φ_0 we will get the frequency drift $\Delta\omega_{\text{local}}$:

$$\Delta\omega_{\text{local}} = \Phi_0 \frac{\dot{\Phi}_{\text{norm}}}{\Phi_{\text{norm}}} - 2\frac{\dot{\Omega}_{t1}}{\Omega_{t1}}\Phi_0. \tag{51}$$

Since

$$\dot{\Phi}_{\text{norm}} = \frac{d\Phi_{\text{norm}}}{d\kappa^2} \frac{d\kappa^2}{dt}, \tag{52}$$

we can write

$$\Delta\omega_{\text{local}} = \Phi_0 \frac{\dot{\Omega}_{t1}}{\Omega_{t1}} \left(\frac{\Phi'_{\text{norm}}}{\Phi_{\text{norm}}} \frac{K(\kappa)}{K'(\kappa)} - 2 \right), \tag{53}$$

where the prime means differentiation with respect to κ^2, and the time derivatives should be taken at a fixed point in space.

It is possible to show that the first term within large parentheses is always larger than 2. Therefore, if the main wave amplitude is increasing, the frequency shift has always the same sign as the phase shift Φ_0.

Figure 5 shows how frequency shifts can be created at a receiver at the end of the interaction region even when Φ_0 is locally constant in time. We assume that electrons propagate in the negative z direction, and for simplicity we reduce the continuous interaction region to three localized points in space where radiation is created. The vertical axes represents the phase of the radiation produced by the resonances if the inhomogeneity were zero. We assume that the wave configu-

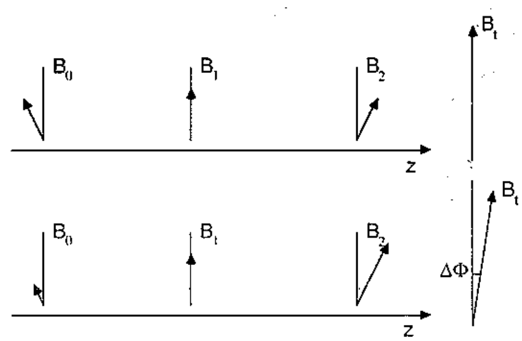

FIG. 5. Schematic representation of the transformation of resonance phase shifts into radiation frequency shifts. B_t is a vector sum of B_0, B_1, and B_2. The vertical axes represent the vector phases if the inhomogeneity were equal to zero. The upper part of the picture shows a situation in which the amplitudes of the component fields add up to a zero-phase-shifted B_t. The lower part shows what happens when component fields vary in magnitude keeping a constant phase. The result is a phase shift on B_t. If the shift $\Delta\Phi$ happens on a time interval Δt, the frequency shift will be $\Delta\omega_{\text{spatial}} \approx \Delta\Phi/\Delta t$.

ration is such as to produce a positive phase shift on the radiation at the beginning of the interaction region (B_2) and therefore a negative shift at its end (B_0). In the middle point there is no inhomogeneity and the shift is zero. We assume that, at a certain moment, B_0 and B_2 have the same amplitude. The phase of the received radiation, B_t, will then be equal to zero. If over a time interval Δt, B_0 decreases and B_2 increases in amplitude, the resulting field, B_t, will undergo a positive phase shift, $\Delta\Phi$. This will be detected by the receiver as an increase in frequency, $\Delta\omega_{\text{spatial}} = \Delta\Phi/\Delta t$. We see that variations in amplitude of component fields having an inhomogeneity-dependent phase shift are enough to create an overall variable frequency field, even when the phase shifts are locally constant in time.

Since the phase shifts are dependent on Ω_{t1}, and therefore on $v_\perp$, we conclude that associated with the shifts in frequency there will be an increase in the linewidths of the sideband waves as they move away from their original positions in the frequency axis.

VI. EXTERNAL RESONANCES—HALF HARMONIC

To study external resonances, we start with the Hamiltonian written as a function of $(\theta,\dot\theta) = (q,p)$:

$$h(q,p,t) = h_0(q,p,t) + \epsilon h_1(q,p,t), \qquad (54)$$

with

$$h_0 = p^2/2,\ h_1 = -\tau q - \sum_{i=1}^{N} A_i \cos(q - \Omega_i t + \phi_i), \qquad (55)$$

where again a factor of ϵ was factored out of τ.

To arrive at the corrections created by the inhomogeneity on the half harmonic resonance, we choose $K_0 = h_0$ and $K_1 = -\tau Q$. This implies

$$W_1 = \sum_{i=1}^{N} \frac{A_i}{(P-\Omega_i)} \sin(Q - \Omega_i t + \phi_i). \qquad (56)$$

The second-order Hamiltonian containing the half harmonic is chosen to be

$$K_2 = \frac{1}{4}\sum_{i=1}^{N} \frac{A_i^2}{(P-\Omega_i)^2} - \frac{A_iA_j}{4}\left(\frac{1}{(P-\Omega_i)^2} + \frac{1}{(P-\Omega_j)^2}\right) \times\cos(2Q - 2\Omega_{1/2}t + \phi_i + \phi_j). \qquad (57)$$

The corrections to the resonance will show up in the K_3 terms depending both on τ and on the argument $(2Q - 2\Omega_{1/2}t + \phi_i + \phi_j)$. To calculate K_3 are need W_2 that can be found similarly to the homogeneous case to be

$$W_2 = \sum_{i,j=1}^{N} \frac{A_iA_j}{2(P-\Omega_i)^2[2P - (\Omega_i + \Omega_j)]} \sin[2Q - (\Omega_i + \Omega_j)t + \phi_i + \phi_j]$$
$$- \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{A_iA_j}{2(P-\Omega_i)^2(\Omega_i - \Omega_j)} \sin[(\Omega_i - \Omega_j)t + (\phi_i - \phi_j)]$$
$$- 2\tau \sum_{i=1}^{N} \frac{A_i}{(P-\Omega_i)^3} \cos(Q - \Omega_i t + \phi_i). \qquad (58)$$

Relevant terms for K_3 will come only from $\frac{1}{2}L_2 h_1$ and from $\frac{1}{2}L_1^2 h_1$. They give

$$\tfrac{1}{2}L_2 h_1 \rightarrow -\frac{3\tau A_iA_j}{2}\left(\frac{1}{(P-\Omega_i)^4} + \frac{1}{(P-\Omega_j)^4}\right) \times\sin(2Q - 2\Omega_{1/2}t + \phi_i + \phi_j) \qquad (59)$$

and

$$\tfrac{1}{2}L_1^2 h_1 \rightarrow \frac{\tau A_iA_j}{2}\left(\frac{(P-\Omega_i)(P-\Omega_j) + (\Omega_i - \Omega_j)^2}{(P-\Omega_i)^3(P-\Omega_j)^3}\right) \times\sin(2Q - 2\Omega_{1/2}t + \phi_i + \phi_j). \qquad (60)$$

K_3 can, therefore, be written as

$$K_3 = -\tau A_iA_j f(P) \sin(2Q - 2\Omega_{1/2}t + \phi_i + \phi_j), \qquad (61)$$

with

$$f(P) = \frac{1}{2}\left(\frac{1}{(P-\Omega_i)^4} + \frac{1}{(P-\Omega_j)^4} - \frac{1}{3}\,\frac{(P-\Omega_i)(P-\Omega_j) + (\Omega_i - \Omega_j)^2}{(P-\Omega_i)^3(P-\Omega_j)^3}\right) \qquad (62)$$

that has the following properties:

$$f(\Omega_{1/2}) = \frac{24}{(\Omega_i - \Omega_j)^4}, \quad f'(\Omega_{1/2}) = 0. \qquad (63)$$

The full Hamiltonian is now

$$K = \frac{P^2}{2} - \epsilon\tau Q + \frac{\epsilon^2}{4}\sum_{i=1}^{N} \frac{A_i^2}{(P-\Omega_i)^2} - \frac{\epsilon^2 A_iA_j}{4}\left(\frac{1}{(P-\Omega_i)^2} + \frac{1}{(P-\Omega_j)^2}\right) \times\cos(2Q - 2\Omega_{1/2}t + \phi_i + \phi_j) - \epsilon^3\tau A_iA_j f(P)\sin(2Q - 2\Omega_{1/2}t + \phi_i + \phi_j). \qquad (64)$$

Defining

$$2\theta_0 = \frac{4\epsilon\tau f(P)(P-\Omega_i)^2(P-\Omega_j)^2}{(P-\Omega_i)^2 + (P-\Omega_j)^2} \approx \frac{12\epsilon\tau}{(\Omega_i - \Omega_j)^2}, \qquad (65)$$

where the approximate equality is valid near the resonance, the Hamiltonian can be rewritten as

$$K = \frac{P^2}{2} - \epsilon\tau Q + \frac{\epsilon^2}{4}\sum_{i=1}^{N} \frac{A_i^2}{(P-\Omega_i)^2} - \frac{\epsilon^2 A_iA_j}{4}\left(\frac{1}{(P-\Omega_i)^2} + \frac{1}{(P-\Omega_j)^2}\right) \times\cos[2(Q-\theta_0) - 2\Omega_{1/2}t + \phi_i + \phi_j]. \qquad (66)$$

We see that apart from creating a small initial phase shift on the equilibrium positions, the inhomogeneity acts on the half harmonic as it does on a regular resonance created directly by a single weak carrier propagating in the magnetosphere. If we calculate the equation of motion, we obtain

$$\ddot{Q} = \epsilon\tau - \frac{4\epsilon^2 A_i A_j}{(\Omega_i - \Omega_j)^2} \times \sin[2(Q-\theta_0) - 2\Omega_{1/2}t + \phi_i + \phi_j] . \quad (67)$$

The equilibrium points are shifted by

$$\Delta Q_{\text{shift}} = \theta_0 + \tfrac{1}{2}\sin^{-1}\left(\frac{\tau(\Omega_i - \Omega_j)^2}{4\epsilon A_i A_j}\right) \quad (68)$$

from their original values. There will be no oscillatory motion if

$$|\tau| > \frac{4\epsilon A_i A_j}{(\Omega_i - \Omega_j)^2} . \quad (69)$$

This is a very small value for the inhomogeneity meaning that the closed orbits associated with such resonances will be easily stripped away and the resonance destroyed as shown in Fig. 2(d).

Frequency shifts and line broadening will also be observed in the radiation coming from external resonances. They are originated in a manner similar to the shifts and broadening associated with internal resonances described in Sec. V, a main difference being that for external resonances any interacting carrier amplitude variation will contribute to the local time dependence of the phase shifts.

VII. HIGHER-ORDER RESONANCES

Higher-order "trapped" and external resonances can also be studied but the expressions describing the effects of the inhomogeneity are extremely complex and can only be arrived at with the help of an algebraic manipulation computer program. Those expressions will not be reproduced here, but a summary of the principal results is possible to make.

A. Internal resonances

Some inhomogeneity effects come directly from the interaction between the main wave and the inhomogeneity and should not depend on the type of resonance under study. Therefore, we should expect to have the same expression for Δj describing the motion of the resonance towards the center of oscillation. The stripping away of the outward orbits of the main wave should remain unaltered, and so should the upward shift of the main resonance when τ is positive. An oscillatory component in the phase shift of the equilibrium points in higher-order resonances should also be present and its mathematical expression should be exactly the same as for first-order resonances, since such a shift is a consequence of the upward displacement of the main wave resonance, and is independent of the particular "trapped" resonance under consideration. The only effect that can and will depend on the type of resonance under study is the value of the constant part of the phase shift, Φ_0. Phase plots such as the one in Fig. 2(b) show that Φ_0 values have all the same order of magnitude, but they do not always have the same sign. A phase plot analysis also shows that the reflection properties of those higher-order phase shifts are the same as for the first-order ones, i.e., they change sign when the frequency offset of all interacting carriers, or the inhomogeneity, is multiplied by -1.

B. External resonances

Higher-order external resonances also have phase shifts proportional to ϵ in the position of their equilibrium points. Those shifts, although all of the same order of magnitude, vary appreciably from one resonance to another and can even be zero as, for example, in the case of the $\frac{1}{3}$ subharmonic. Such shifts are, nevertheless, irrelevant because they are readily swamped by the direct effect of the inhomogeneity on each resonance. An nth-order resonance will have a phase shift of the order of unity, and will be subsequently wiped out when under the influence of an inhomogeneity with a torque $\tau \approx \epsilon^{n-1}$. This is a very small value indeed, and if no appreciable growth is present, those resonances will exist only over a very short interaction region around the equator.

VIII. CONCLUSIONS

The inhomogeneity of the magnetic field of the earth has the following effects on resonances:

(1) For an external resonance, it opens up its outer closed phase orbits, destroying it completely if the associated torque is strong enough, and introduces shifts in the positions of its stable equilibrium points. The resultant shifts have the same sign as the applied torque and, even when locally constant in time, are translated at the end of the interaction region into radiation frequency shifts. Since the phase shifts are dependent on $v_\perp$, the frequency drifts will be accompanied by an increase in the radiated sideband linewidth. Because the external resonances that gives rise to sidebands are naturally narrower than the ones associated with main carriers, they will be able to exist only over a short length of the interaction region located around the equator without being completely destroyed.

(2) For a "trapped" resonance it creates a shift in its angular position inside the main wave and moves its orbit towards the center of the main wave resonance. The shift towards the center is independent of the resonance under consideration, and contributes to extend the lifetime of the resonance by moving it away from orbits easily opened by the inhomogeneity. The phase shifts take values partly dependent on the resonance under consideration, show well-defined parity properties under reflection of the perturbing carriers around the main carrier position, and change sign when the inhomogeneity does. Those shifts, as in the case of external resonances, will be translated into frequency shifts and line broadenings at the receiving end of the interaction region.

The difference in resistance to the inhomogeneity between internal and external resonances is one of the reasons why it is possible to have carriers too weak to be seen in a spectral display creating easily visible sidebands. Since their own resonances are external and narrow, the orbits are opened up, the resonance destroyed, and growth eliminated. The weak carrier is never seen. However, since they can create resonances inside the strong main carrier, and since those resonances are not easily affected by the inhomogeneity, bunching can occur, radiation will be produced, and the sidebands seen.

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