

Current distribution of a VLF electric dipole antenna in the plasmasphere

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[1] In a recent paper (Inan et al., 2003) a method of remediating enhanced energetic electron fluxes in the radiation belt was proposed in which injection of VLF whistler mode waves from spacecraft within the radiation belts would dramatically increase the pitch angle scattering of the relativistic electrons and cause these particles to be rapidly lost from the belts, thereby mitigating the flux enhancement. The VLF wave transmitting system discussed by Inan et al. (2003) involves electric dipole antennas. One of the most important characteristics of such an antenna is the current distribution along the length of the dipole, since it is this current which ultimately determines the amount of VLF power which can be radiated from the antenna into the plasma. In past work it has been assumed without proof that the dipole current has a triangular distribution. In the present work we determine the dipole antenna current distribution from first principles, constructing an integral equation of the Hallén type relating the current distribution to the wave vector potential. In this development it is assumed that the length of the thin cylindrical dipole antenna is small compared to the wavelength of whistler mode waves which propagate parallel to the Earth's magnetic field B_o . In the case of the dipole antenna oriented parallel to B_o , it is found that the assumption of a triangular current distribution is reasonable for antenna lengths up to hundreds of meters. For the case of the antenna perpendicular to B_o , it is found that the current decays exponentially along the antenna from the feed points to the antenna ends. In this case we find the conditions under which a triangular current distribution is still a reasonable approximation. We also give the conditions under which the quasi-static model of Balmain (1964) reasonably describes the electric fields associated with the dipole antenna.

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1. Introduction

[2] One of the important components of space weather is the relativistic electron population of the radiation belts. During periods of magnetic disturbance the enhanced flux of these particles can seriously threaten the growing number of civilian and military assets in space. The vulnerability of these assets continually increases as the trend toward smaller spacecraft results in less radiation protection and the trend toward smaller chips results in less effective radiation hardening.

[3] In a recent paper [Inan et al., 2003], it was proposed that in situ injection of VLF whistler mode

waves from electric dipole antennas on spacecraft within the radiation belts would dramatically increase the pitch angle scattering of the relativistic electrons and cause these particles to be rapidly lost from the belts, thereby mitigating the flux enhancement. In order to assess the number of spacecraft-based VLF transmitters necessary to achieve the proposed mitigation it is necessary to determine the maximum VLF electromagnetic power that can be radiated by a typical spacecraft system using electric dipole antennas. One of the most important factors in this determination is the current distribution along the length of the dipole, since it is this current which ultimately determines the amount of VLF power which can be radiated from the antenna into the plasma.

[4] At the present time there is very little known in general about the current distribution of dipole antennas at VLF frequencies in a magnetoplasma such as the magnetosphere. Much of the past work concerning the

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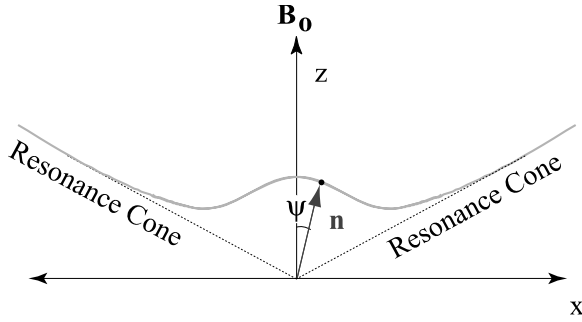


Figure 1. Plot of the whistler mode refractive index as a function of the angle between the k vector and the Earth's magnetic field. The direction of the group velocity for any whistler mode wave propagating at any wave normal angle ψ is given by the direction of the normal to the refractive index surface at that point.

characteristics of dipole antennas in a magnetoplasma has proceeded by first assuming a current distribution, usually triangular, and then determining, for example, the input impedance, radiation resistance, and radiation pattern of the antenna. However, the accuracy of the triangular current assumption has never been established. Numerous references concerning early work on this topic can be found in work by *Balmain* [1972, 1979], *Wang and Bell* [1969], and *Wang* [1970].

[5] In a uniform dielectric medium, one can reasonably approximate the current distribution along a center driven, thin dipole antenna through the relation [*King et al.*, 2002, chapter 1]

$$I(s) \simeq I_o \frac{\sin \beta_d (h - |s|)}{\sin \beta_d h} \quad (1a)$$

where h is the antenna half-length, $\beta_d = \omega \sqrt{\epsilon_d} / c = 2\pi / \lambda$ is the wave number for waves of frequency ω which propagate in the medium, λ is the wavelength of the waves, ϵ_d is the relative dielectric constant, c is the velocity of light in free space, and s is the distance along the antenna measured from the current input terminals at $s = 0$. If the antenna length is small compared to the wavelength of the radiated waves, then $(\beta_d h)^2 \ll 1$ and the sinusoidal terms can be approximated by their arguments and (1a) becomes the triangular current distribution

$$I(s) = I_o \left(1 - \frac{|s|}{h} \right) \quad (1b)$$

[6] If h is fixed in (1a), $I(s)$ depends only upon the unique wavelength $\lambda = 2\pi / \beta_d$ which is possessed by all propagating waves of frequency ω in the medium,

independent of their direction of propagation. This circumstance is quite different in a magnetized plasma such as the magnetosphere, since this medium is anisotropic for electromagnetic waves, and the wavelength of these waves depends strongly upon their direction of propagation. Since there is no unique wavelength in this medium, it is not clear that the current distribution along an in situ dipole antenna can be reasonably described by relations such as (1a) or (1b). Furthermore, if it is possible, what is the value for the wavelength that should be used in (1a)?

[7] The variation of λ at VLF frequencies in the plasmasphere results from the variation of the refractive index $n(\psi)$ as the angle ψ between the propagation vector k and the Earth's magnetic field B_o is changed. In Figure 1 we plot $n(\psi)$ for a typical VLF whistler mode wave (assuming a cold plasma) as a function of ψ . The plot shows a cross section of the refractive index surface, which is a surface of revolution generated by rotating the refractive index plot around the B_o direction. The refractive index n increases monotonically as a function of ψ and extends asymptotically toward infinity along the resonance cone surface at the resonance cone angle $\psi = \psi_r$.

[8] Figure 2 shows $\lambda(\psi)$ for whistler mode waves of 5 kHz frequency whose refractive index is similar to that shown in Figure 1. For waves propagating directly along B_o , $\lambda \simeq 3$ km, while for waves propagating at angles ψ close to ψ_r , $\lambda \sim 0$. The lower limit is an artifact of the assumption that the plasma is cold. Finite temperature effects generally prevent the wavelength from approaching zero. Extensive spacecraft observations of these very short wavelength whistler mode waves, also commonly known as quasi-electrostatic lower hybrid waves, indi-

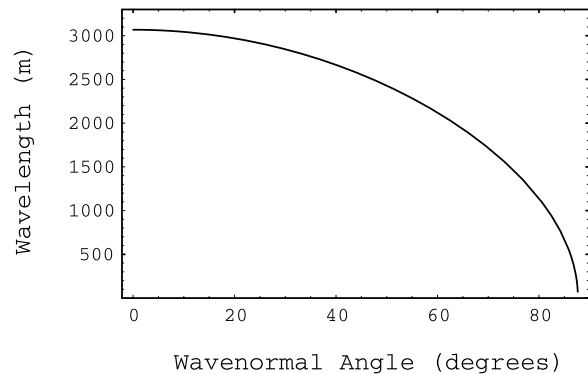


Figure 2. Wavelength of a whistler mode wave of 5 kHz frequency as a function of the wavenormal angle of the wave. In preparing this plot, it was assumed that the location was near the magnetic equatorial plane at an altitude of 6000 km, where $f_o = 450$ kHz and $f_{ce} = 100$ kHz.

cate that their wavelengths are seldom smaller than $\simeq 2$ m [Bell *et al.*, 1983; Bell and Ngo, 1988; James and Bell, 1987; Bell and Ngo, 1990; Bell *et al.*, 1991a, 1991b, 1994]. Since the dipole antenna length proposed by Inan *et al.* [2003] was roughly 100 m, we have a situation in which some of the waves of frequency ω radiated by the antenna will possess wavelengths much longer than the antenna while other radiated waves of frequency ω will possess wavelengths much shorter than the antenna. In view of the complexity of this system, some method of determining $I(s)$ from first principles is clearly required.

[9] Our primary goal in the present paper is to construct from first principles integral equations of the Hallén type [Hallén, 1938] to determine $I(s)$ for two orientations of the dipole antenna, one parallel to B_o and one perpendicular to B_o . These equations are developed in the limit of short wavelengths, ie, for $\lambda^2 \ll \lambda_o^2$, where λ_o is the wavelength of whistler mode waves propagating parallel to B_o . This approach is suggested by the results of Wang and Bell [1969], who assumed a current distribution similar to (1b) and found that the major portion of the radiated power was carried by quasi-electrostatic whistler mode waves for which $\lambda \simeq h$. This result suggests that if the antenna length is restricted to 100 m or less, one can reasonably neglect the effects of the waves with $\lambda \geq 1$ km and consider only the shorter wavelength quasi-electrostatic waves. Our second goal is to determine the conditions under which $I(s)$ is approximately triangular in order to judge the applicability of past work [e.g., Balmain, 1964; Wang and Bell, 1969; Wang, 1970] in which such distributions were assumed to apply.

[10] Our third goal is to compare the results of our model with the results of past workers who used both electromagnetic and quasi-static models to describe the fields generated by a dipole antenna. The quasi-static model is based on a modified form of Poisson's equation, and Balmain [1964] was one of the first to apply the model to a thin electric dipole in a magnetized plasma. He assumed that the antenna possessed a triangular current distribution and was short compared to the characteristic wavelength of the medium. This characteristic wavelength is not defined in the model, but must be determined through other means [Chugunov, 1968]. With the aid of the quasi-static model, the input impedance of the antenna was defined through concise analytic formulas, and the predictions of the model were partially tested in laboratory experiments [Balmain, 1964].

[11] Subsequently, Wang and Bell [1969] and Wang [1970] studied the same problem with the aim of establishing the general characteristics of dipole antennas in the plasmasphere. In contrast to the approximate quasi-static model of Balmain [1964], Wang's model relied on a numerical solution of the full set of Maxwell's equations to find the antenna input impedance, assuming

a triangular current distribution. Comparison of the full wave model with the quasi-static model generally showed good agreement when the antenna length was of the order of 10–100 m and the driving frequency lay in the range: $f_{lhr} \ll f \ll f_{ce}$, where f_{lhr} is the lower hybrid resonance frequency, defined below, and f_{ce} is the electron gyrofrequency.

[12] Notable advances in the development of the quasi-static model were achieved by Chugunov [1968], who considered a wider variety of antenna forms and developed an integral equation relating the integral of the unknown antenna surface charge density to known functions. More recently, Mareev and Chugunov [1987] have extended the model further by incorporated spatial dispersion and particle collisions.

[13] It is important to note here that we do not use the quasi-static model in the present paper. In contrast to the quasi-static model which makes use of a modified form of Poisson's equation, our own model makes use of the full set of Maxwell's equations to construct integral equations for the dipole current distribution. One of the additional outcomes of our development is that we can determine the conditions for the applicability of the quasi-static model in the two cases in which the antenna is either parallel or perpendicular to the ambient magnetic field.

2. Model

[14] Assuming a cold plasma, the electromagnetic fields produced by an electric dipole antenna in the plasmasphere operating at the frequency ω can be characterized through Maxwell's equations:

$$\nabla \times E(r) = -i\omega B(r) \quad (2)$$

$$\nabla \times B(r) = \frac{i\omega}{c^2} \bar{K}E(r) + \mu_o J_a(r) \quad (3a)$$

where $E(r)$ and $B(r)$ are, respectively, the vector electric and magnetic fields produced by the vector electric dipole current $J_a(r)$, r is the position vector of the observation point, and the dielectric tensor \bar{K} has the value

$$\bar{K} = \begin{pmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \quad (3b)$$

where the dielectric constants S , D , and P are the plasma parameters defined by Stix [1962, p. 10] for a cold multicomponent plasma. An important feature of the tensor \bar{K} is the fact that the diagonal dielectric constant P is always negative for the frequency range considered by Inan *et al.* [2003]. This insures that a refractive index

resonance cone will exist as long as the other diagonal dielectric constant S is positive.

[15] For VLF frequencies at locations within the radiation belts the components of \bar{K} can be simply related to the local plasma parameters

$$\begin{aligned} S &\simeq \frac{f_o^2}{f_{ce}^2} \left(1 - \frac{f_{thr}^2}{f^2} \right) \\ D &\simeq \frac{f_o^2}{ff_{ce}} \\ P &\simeq -\frac{f_o^2}{f^2} \end{aligned} \quad (4)$$

where f_o is the local plasma frequency, f_{ce} is the local electron gyrofrequency, f is the wave frequency, $f_{thr} \simeq \sqrt{f_{ce}f_{cp}}$ is the local lower hybrid resonance frequency, f_{cp} is the local proton gyrofrequency, and it is assumed that $f^2 \ll f_{ce}^2$. In the frequency range of interest the dielectric constants can be ordered according to magnitude:

$$|P| \gg D \gg S$$

[16] It is useful to introduce the vector potential function $A(r)$ and the scalar potential $\phi(r)$ through which B and E can be defined:

$$B(r) = \nabla \times A(r) \quad (5)$$

$$E(r) = -\nabla\phi(r) - i\omega A(r) \quad (6)$$

[17] Using (5) and (6) in conjunction with (2) and (3) we arrive at the following relations for the potential functions in terms of the source functions:

$$\begin{aligned} \nabla^2 A(r) + \beta^2 \bar{K} A(r) - \nabla(\nabla \cdot A(r)) \\ - \frac{i\omega}{c^2} \bar{K} \nabla\phi(r) = -\mu_o J_a(r) \end{aligned} \quad (7)$$

$$\nabla \cdot \bar{K} \nabla\phi(r) + i\omega \nabla \cdot \bar{K} A(r) = -\rho_a(r)/\epsilon_o \quad (8)$$

where $\rho_a(r)$ is the charge that appears along the antenna as a result of spatial variations in $J_a(r)$.

[18] We wish to find the solution of (7) and (8) in order to set up an integral equation from which we can find the current distribution $I(s)$ along the antenna. However the solution of these two equations in configuration space is difficult because they are coupled second-order partial differential equations whose solution requires the solution of a fourth-order partial differential equation. However, considerable simplification of the equations can be achieved when they are formulated in the short wavelength limit, as defined below. We apply this limit only

after we have found the general Fourier k space solutions to (7) and (8).

[19] To proceed we define the divergence of A with the following generalized Lorentz condition:

$$\nabla \cdot A(r) = -\frac{i\omega C_1}{c^2} \phi \quad (9)$$

where C_1 can be interpreted as an effective relative dielectric constant. Since only the curl of $A(r)$ is defined through Maxwell's equations, the $E(r)$ and $B(r)$ fields do not depend upon C_1 . Thus the choice of a value for C_1 is arbitrary. However judicious choice of a value for C_1 can simplify the relations used to find $A(r)$.

[20] It should be noted here that although the relationship given in (9) is useful in the present development, other relationships may be more efficacious for determining the scalar potential $\phi(r)$. For example, in the place of (9), *Balmain* [1964] imposed the condition $\nabla \cdot \bar{K} A(r) = 0$ in (8) in order to directly find $\phi(r)$ for the case of a short dipole antenna with a triangular current distribution. This model has been recently been applied to the case of quasi-electrostatic waves emitted by dipole antennas in the ionosphere [*James*, 2000; *Chugunov*, 2001], as well as the case of Z mode radiation observed during rocket experiments involving the transmission and reception of VLF whistler mode waves and HF waves [*Chugunov et al.*, 2003; *James*, 2004].

[21] Now eliminating $\phi(r)$ from (7) with the use of (9) we obtain

$$\begin{aligned} \nabla^2 A(r) + \beta^2 \bar{K} A(r) - \nabla(\nabla \cdot A(r)) \\ + \bar{K} \nabla(\nabla \cdot A(r))/C_1 = -\mu_o J_a(r) \end{aligned} \quad (10)$$

Applying a spatial Fourier transform to (10), the equations for the three scalar components of $A(r)$, $A_x(r)$, $A_y(r)$, and $A_z(r)$ can be obtained in complex notation:

$$\begin{aligned} A_+(k) + \frac{k_+(k \cdot A(k))(R/C_1 - 1)}{(k^2 - \beta^2 R)} &= \frac{\delta_+}{(k^2 - \beta^2 R)} \\ A_-(k) + \frac{k_-(k \cdot A(k))(L/C_1 - 1)}{(k^2 - \beta^2 L)} &= \frac{\delta_-}{(k^2 - \beta^2 L)} \\ A_z(k) + \frac{k_z(k \cdot A(k))(P/C_1 - 1)}{(k^2 - \beta^2 P)} &= \frac{\delta_z}{(k^2 - \beta^2 P)} \end{aligned} \quad (11)$$

where R , L , and P are parameters defined by *Stix* [1962, 1992], $\beta = \frac{\omega}{c}$, $\delta_z = \mu_o J_{az}(k)$, $A_\nu(k) = A_x(k) + i\nu A_y(k)$, $k_\nu = k_x + i\nu k_y$, $\delta_\nu = \mu_o (J_{ax}(k) + i\nu J_{ay}(k))$, where $\nu = +1$, or -1 .

[22] With the aid of (11) we can find an expression for $k \cdot A(k)$:

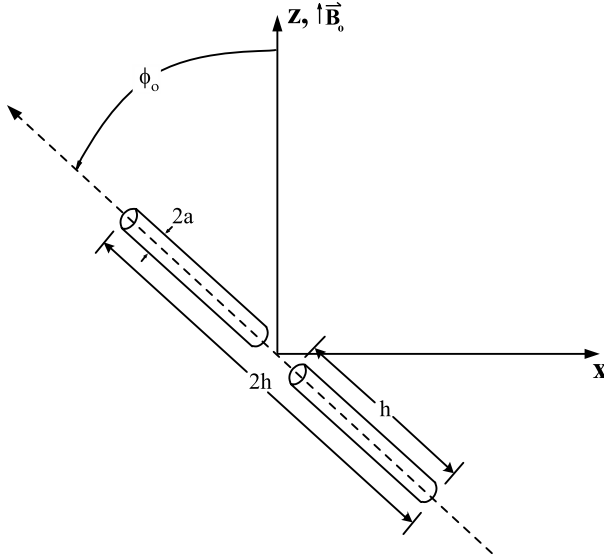


Figure 3. Sketch of a symmetric center-driven linear dipole antenna.

$$k \cdot A(k) = \frac{C_1 k^2}{g(k)} \left(\frac{k_{-1} \delta_{+1}}{2(k^2 - \beta^2 R)} + \frac{k_{+1} \delta_{-1}}{2(k^2 - \beta^2 L)} + \frac{k_z \delta_z}{k^2 - \beta^2 P} \right) \quad (12)$$

where

$$g(k) = (k^2 - \beta^2 C_1) \left(\frac{k_{\perp}^2 R}{2(k^2 - \beta^2 R)} + \frac{k_{\perp}^2 L}{2(k^2 - \beta^2 L)} + \frac{k_z^2 P}{k^2 - \beta^2 P} \right)$$

and $k_{\perp}^2 = k_x^2 + k_y^2$. Equations (11) and (12) represent the general solution for $A(k)$ in terms of the given sources. The solution for $A(r)$ in configuration space can be directly obtained through a 3-D inverse Fourier transform of $A(k)$.

[23] We are interested in the case of short wavelength waves whose wave normals are close to the resonance cone. For these waves, $k^2/\beta^2 \gg |R|, |L|, |P|$, and $|C_1|$. With this in mind, we treat the quantity β^2 as a small expansion parameter and express $g(k)$ as a two term power series in β^2 :

$$g(k) \simeq S k_{\perp}^2 + P k_z^2 - P \beta_c^2 \quad (13)$$

where $\beta_c^2 = \beta^2 [(RL \frac{k_{\perp}^2}{k^2} + SP(1 + \frac{k_z^2}{k^2})]/P + C_1(S \frac{k_{\perp}^2}{k^2} + P \frac{k_z^2}{k^2})]$. The dispersion relationship for the short wavelength whistler mode waves can be obtained by setting $g(k) = 0$

in (13). We carry two terms in the expansion of $g(k)$ in order to provide a propagation constant β_c for these waves. However, in evaluating the remaining expressions it is sufficient to take the leading terms as $\beta \rightarrow 0$. These actions lead to the final equation set:

$$A_+(k) = \frac{\delta_+}{k^2} - \frac{k_+(k \cdot A(k))(R/C_1 - 1)}{k^2} \quad (14a)$$

$$A_-(k) = \frac{\delta_-}{k^2} - \frac{k_-(k \cdot A(k))(L/C_1 - 1)}{k^2} \quad (14b)$$

$$A_z(k) = \frac{\delta_z}{k^2} - \frac{k_z(k \cdot A(k))(P/C_1 - 1)}{k^2} \quad (14c)$$

$$k \cdot A(k) = C_1 \frac{\frac{1}{2}(k_{-1} \delta_{+1} + k_{+1} \delta_{-1}) + k_z \delta_z}{S k_{\perp}^2 + P k_z^2 - P \beta_c^2} \quad (14d)$$

For simplicity we approximate β_c by its value at the resonance cone angle where $\frac{k_x^2}{k^2} = \frac{P}{P-S}$, $\frac{k_z^2}{k^2} = \frac{S}{S-P}$, and $\beta_c^2 \simeq \beta^2 (RL + PS - 2S^2)/(P - S)$. In terms of the plasma parameters f_o and f_{ce} , it can be shown that over the range of frequencies of interest, $(RL + PS - 2S^2)/(P - S) \simeq 2f_o^2/f_{ce}^2$. Thus $\beta_c^2 \simeq 2f_o^2/f_{ce}^2 \beta^2$. Since typically $f_o \simeq 450$ kHz and $f_{ce} \simeq 100$ kHz, we have typically $\beta_c^2 \simeq 40\beta^2$.

2.1. Integral Equation for Current of Antenna Parallel to B_o

[24] Figure 3 shows a sketch of the antenna type considered in the present paper, a symmetric center-driven linear electric dipole antenna of total length $2h$ and radius a . In Figure 3 the antenna axis is assumed to lie in the x - z plane, and the angle ϕ_o represents the angle between B_o , assumed to lie along the z axis, and the antenna axis. In the present work we consider the two cases in which $\phi_o = 0$ or $\phi_o = \pi/2$. When the antenna is parallel to B_o ($\phi_o = 0$), the only component of $J_a(r)$ is $J_z(r)$. In this case it is convenient to choose $C_1 = S = \frac{1}{2}(R+L)$, for which the expressions for $k \cdot A(k)$ and $A_z(k)$ become

$$k \cdot A(k) = \frac{\mu_o k_z J_z(k) S}{S k_{\perp}^2 + P k_z^2 - P \beta_c^2} \quad (15)$$

and

$$A_z(k) = \frac{\mu_o J_z(k) S}{S k_{\perp}^2 + P k_z^2 - P \beta_c^2} \quad (16)$$

where terms of order β_c^2/k^2 in the numerator of (16) have been neglected.

[25] To find the Green's function solution for (16), we introduce the unit source current:

$$J_z(r, r') = \delta(x - x')\delta(y - y')\delta(z - z') \quad (17)$$

where x' , y' , and, z' are the spatial coordinates of the unit source current. The spatial Fourier transform of (17) has the value

$$J_z(k) = e^{i(k_x x' + k_y y' + k_z z')} \quad (18)$$

[26] Using (18) in (16), the inverse Fourier transform of (16) is readily performed to yield the Green's function solution for A_z :

$$A_{zg}(r, r') = \frac{\mu_o}{4\pi} \frac{e^{-i\beta_c R_g}}{R_g} \quad (19a)$$

where $R_g = [-\gamma^2(x - x')^2 - \gamma^2(y - y')^2 + (z - z')^2]^{1/2}$, r is the position vector of the observation point, r' is the position vector of the unit source current, and $\gamma^2 = |\frac{\beta}{S}|$. In (19a) it is assumed that R_a is real, in which case the relation satisfies the boundary condition that all waves are outgoing from the source region. In the event that R_a is purely imaginary, the correct boundary condition is that the evanescent fields vanish at large distances from the source. In this case the Green's function has the form

$$A_{zg}(r, r') = \frac{i\mu_o}{4\pi} \frac{e^{-\beta_c |R_g|}}{|R_g|} \quad (19b)$$

where $|R_a|$ is the absolute value of R_a . With the aid of (19), we can now find A_z in terms of the unknown antenna current:

$$A_z(r) = \frac{\mu_o}{4\pi} \int \frac{e^{-i\beta_c R_g}}{R_g} J_z(z') ds' \quad (20)$$

where $J_z(z')$ is the antenna surface current density, ds' is a surface element containing the antenna current, and it is understood that the Green's function in (20) will be (19b) whenever R_a is purely imaginary.

[27] We now wish to use (20) to construct an integral equation of the type developed by *Hallén* [1938]. This type of integral equation is discussed in detail in a number of textbooks [e.g., *Kraus*, 1950; *King et al.*, 2002]. In order to do this, we first assume that the antenna is cylindrical and thin, with a length that is much larger than the radius a of the cylinder, ie, $h \ll a$. This restriction allows us to neglect charge which may reside on the antenna ends. To proceed we apply the boundary condition that $E_z = 0$ on the surface of the

antenna so that $I(z)$ can be linked to $A_z(r)$. We can find $E_z(r)$ from (6):

$$E_z(r) = -\frac{\partial\phi(r)}{\partial z} - i\omega A_z(r) \quad (21)$$

[28] In addition, the inverse Fourier transform of (14d) can be readily performed to yield the relation

$$\nabla \cdot A(r) = \frac{\partial A_z(r)}{\partial z}$$

[29] Making use of the above relations, E_z can be expressed in terms of A_z alone:

$$\frac{i\omega S}{c^2} E_z(r) = \frac{\partial^2 A_z(r)}{\partial z^2} + \beta_s^2 A_z(r) \quad (22a)$$

where $\beta_s^2 = \beta^2 S$. On the surface of the antenna, $E_z = 0$, except in the gap between the antenna elements where the driving voltage V_o is applied. Thus (22a) can be written

$$-\frac{i\omega S}{c^2} V_o \delta(z) = \frac{\partial^2 A_{za}(r)}{\partial z^2} + \beta_s^2 A_{za}(r) \quad (22b)$$

where $A_{za}(r)$ is the value of $A_z(r)$ on the antenna surface. If we now assume that $A_{za}(r)$ is cylindrically symmetric about the z axis, it will be a function of z alone and the solution to (22b) is

$$A_{za}(z) = -i\frac{\beta_s}{\omega} \left(b_o \cos\beta_s z + \frac{1}{2} V_o \sin\beta_s |z| \right) \quad (23)$$

where in (23) the symmetry condition has been imposed: $A_{za}(z) = A_{za}(-z)$.

[30] The next step is to evaluate (20) on the antenna surface. For simplicity we assume that the source points are located along the z' axis, while the observation points are located on the wire surface. This approximation simplifies the integral equation and is commonly used in constructing integral equations for the current flow along dipole antennas in air [*King et al.*, 2002]. On the antenna surface we let $x = a \sin \alpha$ and $y = a \cos \alpha$ and assume that the surface current density $J_z(z')$ is uniformly distributed around the wire, and thus independent of α at any point z' . With these actions (20) becomes

$$A_{za}(z) = \frac{\mu_o}{4\pi} \int_{-h}^h \frac{e^{-i\beta_c R_a(z, z')}}{R_a(z, z')} I(z') dz' \quad (24)$$

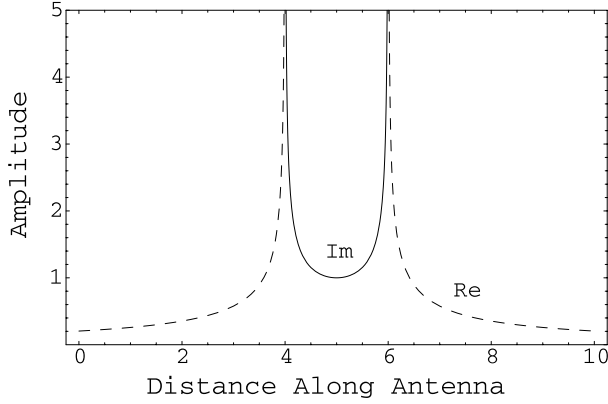


Figure 4. Amplitude of Green's function as a function of the normalized distance along the antenna.

where $R_a(z, z') = [-\gamma^2 a^2 + (z - z')^2]^{1/2}$, a is the antenna radius, and h is the antenna half-length.

[31] We can now combine (23) and (24) to form an integral equation for the antenna current:

$$\frac{\mu_o}{4\pi} \int_{-h}^h \frac{e^{-i\beta_c R_a(z, z')}}{R_a(z, z')} I(z') dz' = A_{za}(z) \quad (25)$$

where b_o is a constant which is determined from the condition that the current vanishes at the antenna end points and the quantity $A_{za}(z)$ is defined in (23).

[32] It is clear from the definition of $R_a(z, z')$ that the integrand on the left hand side of (25) becomes arbitrarily large for any given value of z when the condition obtains:

$$z' = z \pm a\gamma \quad (26)$$

[33] This singularity occurs because of the resonance cone in the whistler mode refractive index (see Figure 1). All the waves whose wave vectors k lie near the resonance cone will propagate in the direction perpendicular to the resonance cone. This direction, referred to as the z axis, is the direction indicated by (26).

[34] To proceed, we form a difference kernel, $K_d(z, z')$, in (25) by subtracting from both sides the integral:

$$H(h) = \frac{\mu_o}{4\pi} \int_{-h}^h \frac{e^{-i\beta_c R_a(h, z')}}{R_a(h, z')} I(z') dz'$$

[35] This action results in the expression

$$\frac{\mu_o}{4\pi} \int_{-h}^h K_d(z, z') I(z') dz' = A_{za}(z) - H(h) \quad (27a)$$

where

$$K_d(z, z') = \frac{e^{-i\beta_c R_a(z, z')}}{R_a(z, z')} - \frac{e^{-i\beta_c R_a(h, z')}}{R_a(h, z')} \quad (27b)$$

[36] The left hand side of (27a) is identically equal to 0 when $z = h$. This boundary condition allows us to solve for the constant b_o contained in $A_{za}(z)$ in terms of H and V_o . This leads to the final equation for the current distribution:

$$\frac{\mu_o}{4\pi} \int_{-h}^h K_d(z, z') I(z') dz' = i \frac{V_o \beta_s}{2\omega} \frac{\sin \beta_s (h - |z|)}{\cos \beta_s h} - \left(1 - \frac{\cos \beta_s z}{\cos \beta_s h}\right) H(h) \quad (28)$$

[37] The most important characteristics of the difference kernel $K_d(z, z')$, as given in (27b) and (28), arise from the first term: $e^{-i\beta_c R_a(z, z')}/R_a(z, z')$. The denominator of this term is purely real when $|z - z'| \geq a\gamma$ and purely imaginary when $|z - z'| \leq a\gamma$. At points where $|z - z'| = a\gamma$, the term possesses integrable singularities. For typical plasmaspheric applications, $\gamma \sim 10-50$, the dipole antenna wire radius generally lies in the range $a \sim 1$ mm to 1 cm, and the antenna half-length lies in the range $h \sim 20-100$ m. Thus for a given value of z , $R_a(z, z')$ will be imaginary only over a distance $\Delta z'$ along the antenna of at most ~ 50 cm, which is small compared to h .

[38] The term $R_a(z, z')^{-1}$ is plotted in Figure 4. The unit distance Δs along the antenna is $\Delta s = a\gamma$. It is assumed that $z = 5$. The function is purely imaginary over the small range: $1 \geq |z' - 5|$, and purely real elsewhere. It can be seen that this term is large only in the regions where $z \simeq z'$. Thus to first order $R_a(z, z')^{-1}$ resembles the delta function, $C_o \delta(z - z')$, where C_o is a complex constant. In this case the distribution of $I(z')$ along the antenna will closely resemble the distribution given on the right hand side of (28). This result is interesting since it suggests that the current distribution is predominantly determined by the diagonal dielectric constant S in the dielectric tensor \bar{K} , since $\beta_s = \omega\sqrt{S}/c$.

[39] Equation (28) can be employed in general to find $I(z)$ using known methods [King *et al.*, 2002], once the values of β_c , β_s , γ , h , and ω are specified. Thus our primary goal of finding a method to determine $I(z)$ from first principles has been achieved.

[40] An equation similar to (25) was derived by Chugunov [1969] for the case of a thin cylindrical dipole antenna in a general uniaxial medium. The magnetospheric plasma has the characteristics of a uniaxial medium at a finite number of frequencies below the proton gyrofrequency where the Stix [1992] plasma parameter D vanishes. In this case the only finite

components of the dielectric tensor \bar{K} are the diagonal terms S and P . In general it can be expected that the uniaxial approximation would be reasonable if $|D| \ll |S|$ and $|D| \ll |P|$. The uniaxial approximation does not directly apply in the present work, because over the frequency range of interest $|S| \ll |D|$. However, the fact that (25) is similar to the integral equation of *Chugunov* [1969] suggests that in the short wavelength limit the magnetospheric plasma resembles a uniaxial medium for a dipole antenna oriented parallel to B_o .

2.1.1. Applicability of Triangular Current Distribution and Quasi-Static Model

[41] To fulfill our secondary goal, we can now make use of (28) to determine the conditions under which the current distribution can be approximated by a triangular current similar to (1b). It is clear that the right hand side of (28) can be reduced to a triangular distribution in z whenever $(\beta_s h)^2 \ll 1$. If we assume that $f_o = 450$ kHz, $f_{ce} = 100$ kHz, and $f = 5$ kHz, then $\beta_s h \simeq 5h \cdot 10^{-4}$. A choice of $h = 200$ m results in the value $(\beta_s h)^2 \simeq 10^{-2}$. Thus the choice of a triangular current distribution for the antenna would appear to be reasonable up to antenna total lengths of at least $2h \simeq 400$ m. A triangular distribution may also apply to even greater lengths; however it must be kept in mind that our derivation of (28) was carried out under the assumption that the antenna is small compared to the longest wavelengths of the waves radiated by the antenna, which are approximately 3 km when $f \simeq 5$ kHz.

[42] We can also use (28) to find the conditions under which quasi-static models, such as that of *Balmain* [1964], can reasonably be used to describe the electric field of the antenna. These models should be applicable as long as the wave phase variation along the antenna is small; ie, when $(\beta_c R_a)^2 \ll 1$ in (27b) and (28). In most practical cases we expect that $a^2 \gamma^2 \ll h^2$. In this case the maximum value of $R_a(z, z')$ is $R_a(z, z') = 2h$. As shown previously, $\beta_c^2 \simeq 2\beta_o^2 f_o^2 / f_{ce}^2$. Thus the restriction can be written: $8h^2 \beta_o^2 f_o^2 / f_{ce}^2 \ll 1$. The maximum value of the ratio f_o / f_{ce} occurs in the magnetic equatorial plane near $L \simeq 4$ where $f_o / f_{ce} \simeq 14$. If we now assume $f = 5$ kHz and $h = 50$ m, we have $8h^2 \beta_o^2 f_o^2 / f_{ce}^2 \simeq 0.04$. Thus the quasi-static model should apply well for this particular antenna orientation.

2.1.2. Antenna Input Impedance

[43] If (28) is valid, it should yield the correct input impedance of the antenna. To test this relationship, we can compare the predictions of (28) with the results of *Balmain* [1964], *Wang and Bell* [1969], and *Wang* [1970]. In these studies a triangular current distribution was assumed, implying $(\beta_s h)^2 \ll 1$, and we have seen above that $(\beta_c R_a)^2 \ll 1$ for typical antenna lengths. Applying these limits to (28), we arrive at the relation

$$\frac{\mu_o}{4\pi} \int_{-h}^h K_d(z, z') I(z') dz' = i \frac{V_o \beta_s}{2\omega} \beta_s (h - |z|) \quad (29)$$

where now

$$K_d(z, z') \simeq \frac{1}{R_a(z, z')} - \frac{1}{R_a(h, z')} \quad (30)$$

[44] Following the procedure outlined by *King et al.* [2002, chapter 2], it can be shown that the first-order solution to (29) is given by the expression

$$I(z) = \frac{2\pi i V_o}{Z_o \Psi} \beta_s (h - |z|) \quad (31a)$$

where $Z_o = \sqrt{\mu_o / \epsilon_o}$ is the impedance of free space and

$$\Psi = \int_{-h}^h \left(1 - \frac{|z'|}{h}\right) K_d(0, z') dz' \quad (31b)$$

[45] Carrying out the straightforward integration in (31b), it is found that

$$\Psi = \pi i + 2 \left[\log \left(\frac{2h}{\gamma a} \right) - 1 \right] \quad (31c)$$

where it is assumed that $\log \left(\frac{2h}{\gamma a} \right) \gg 1$.

[46] From the preceding we can determine the antenna input impedance for the parallel antenna:

$$Z_{in} = V_o / I(0) = R_r + iX_{||}$$

where R_r is the radiation resistance of the antenna and X is the reactance of the antenna, given by the expressions

$$R_r = \frac{Z_o}{2\beta_s h S} \quad (32a)$$

$$X_{||} = -\frac{Z_o}{\pi\beta_s h S} [\log(2h/\gamma a) - 1] \quad (32b)$$

[47] As discussed in section 3, (32a) and (32b) are identical to the leading term of the value for Z_{in} obtained by the full wave solution for a triangular current distribution. In addition, they also agree well with the predictions of the quasi-static model [*Balmain*, 1964]. This gives confidence that the integral equation (28) derived in the short wavelength limit contains the essential physics of the radiation process.

2.2. Integral Equation for Current of Antenna Perpendicular to B_o

[48] As a second example we consider a dipole antenna oriented along the x axis, perpendicular to B_o , with current $J_x(r)$. It can be seen from Figure 1 that whistler mode waves in the frequency range of interest cannot

propagate directly perpendicular to B_o . However, the resonance cone angle for $f \sim 5$ kHz waves is $\psi_r \sim 87^\circ$. Thus waves can propagate in the medium with their wave normals very close to the perpendicular direction, and it is not clear if the current flow along the antenna will take the form of propagating waves or evanescent waves.

[49] To investigate this case, we choose $C_1 = 2(P - S)$ and then drop terms of order R/P and L/P in (11) on the grounds that in magnetospheric applications at VLF frequencies, $|R/P|, |L/P| \ll 1$. The expressions for $A_x(k)$, $A_y(k)$, $A_z(k)$, and $k \cdot A(k)$ then become

$$A_x(k) = \frac{\mu_o J_x(k)}{k^2} \left[1 + \frac{2k_x^2(P - S)}{Sk_\perp^2 + Pk_z^2 - P\beta_c^2} \right] \quad (33a)$$

$$A_y(k) = \frac{\mu_o J_x(k)}{k^2} \frac{2(P - S)k_x k_y}{Sk_\perp^2 + Pk_z^2 - P\beta_c^2} \quad (33b)$$

$$A_z(k) = \frac{\mu_o J_x(k)}{k^2} \frac{(2S - P)k_x k_z}{Sk_\perp^2 + Pk_z^2 - P\beta_c^2} \quad (33c)$$

$$k \cdot A(k) = \mu_o J_x(k) \frac{2k_x(P - S)}{Sk_\perp^2 + Pk_z^2 - P\beta_c^2} \quad (33d)$$

[50] We note that in (33), $k_x = k_\perp \cos \theta$ and $k_y = k_\perp \sin \theta$. The RMS value of k_x^2 can be found by averaging over θ to yield $\langle k_x^2 \rangle_{rms} = \frac{1}{2} k_\perp^2$. For simplicity, in (33a) we replace k_x^2 by its RMS value and group terms to arrive at the result

$$A_x(k) = \mu_o J_x(k) \frac{P}{Sk_\perp^2 + Pk_z^2 - P\beta_c^2} \quad (34)$$

where terms of order β_c^2/k^2 have been neglected in the numerator.

[51] In addition to (34), we will need the three following equations:

$$E_x(r) = -\frac{\partial \phi(r)}{\partial x} - i\omega A_x(r) \quad (35a)$$

$$\nabla \cdot A(r) = -\frac{2i\omega(P - S)}{c^2} \phi \quad (35b)$$

$$\nabla \cdot A(r) = 2(1 - S/P) \frac{\partial A_x(r)}{\partial x} \quad (35c)$$

where (35c) can be deduced from (33d) and (34).

[52] Using (34) and (35) we can express $E_x(r)$ in terms of $A_x(r)$:

$$-\frac{i\beta_p^2}{\omega} E_x(r_a) = \frac{\partial^2}{\partial x^2} A_x(r_a) - \beta_p^2 A_x(r_a) \quad (36)$$

where $\beta_p^2 = \beta^2|P|$ and the position vector r_a represents points on the antenna surface. We assume that $A_x(r_a)$ is symmetric about the x axis and thus is a function of x alone. In this case since on the surface of the antenna we have $E_x(x) = -V_o \delta(x)$, the solution of (36) takes the form

$$A_x(x) = -\frac{i\beta_p}{\omega} \left(b_o \cosh \beta_p x - \frac{1}{2} V_o \sinh \beta_p |x| \right) \quad (37)$$

[53] If we now define a unit source current along the x axis equal to the right hand side of (18), the inverse Fourier transform of $A_x(k)$ can be readily performed to yield the Green's function solution for $A_x(r)$:

$$A_{xg}(r, r') = \frac{\mu_o P}{4\pi S} \frac{e^{-i\beta_c R_g}}{R_g} \quad (38a)$$

where it is assumed that R_g is purely real. When R_g is purely imaginary, the Green's function takes the form

$$A_{xg}(r, r') = \frac{i\mu_o P}{4\pi S} \frac{e^{-\beta_c |R_g|}}{|R_g|} \quad (38b)$$

where $|R_g|$ is the absolute value of R_g .

[54] It can be seen that the Green's function for A_x is linearly proportional to the Green's function for A_z given in (19). However the actual values for R_g are quite different. With the aid of (38), an expression for $A_x(r)$ can be found in terms of the unknown antenna surface current:

$$A_x(r) = \frac{\mu_o P}{4\pi S} \int \frac{e^{-i\beta_c R_g}}{R_g} J_x(r') ds' \quad (39)$$

where it is understood that the Green's function in (39) will be (38b) whenever R_g is purely imaginary.

[55] To evaluate $A_x(r)$ on the surface of the perfectly conducting antenna wire oriented along the x axis, we introduce polar coordinates in which the polar angle is measured about the x axis and $y = a \sin \phi$ and $z = a \cos \phi$. In this system we find

$$R_g = \left[-\gamma^2 (x - x')^2 - \gamma^2 (a \sin \phi)^2 + (a \cos \phi)^2 \right]^{1/2} \quad (40)$$

[56] Since $\gamma \gg 1$, it can be seen that R_g will be purely imaginary for almost all values of $|x - x'|$ except for a narrow range of values for ϕ and $|x - x'|$ for which

$$\frac{a^2 \cos^2 \phi}{\gamma^2} \geq a^2 \sin^2 \phi + (x - x')^2$$

[57] Taking into account the fact that R_g is primarily purely imaginary, the expression for $A_x(r)$ can be written

$$A_x(r) = -i \frac{\mu_o \gamma}{8\pi^2} \int_0^{2\pi} \int_{-h}^h \frac{e^{-\beta_c \gamma R_a}}{R_a} J_x(x') dx' d\phi \quad (41)$$

where

$$R_a = \left[(x - x')^2 + (a \sin \phi)^2 - (a \cos \phi)^2 / \gamma^2 \right]^{1/2}$$

[58] It can be seen that the Green's function in (41) now represents, for the most part, evanescent waves rather than propagating waves.

[59] Combining (41) and (37) we arrive at the integral equation

$$\begin{aligned} \frac{\mu_o \gamma}{8\pi^2} \int_0^{2\pi} \int_{-h}^h \frac{e^{-\beta_c \gamma R_a}}{R_a} I(x') dx' d\phi \\ = \frac{\beta_p}{\omega} \left(b_o \cosh \beta_p x - \frac{1}{2} V_o \sinh \beta_p |x| \right) \end{aligned} \quad (42)$$

[60] From the form of (42) it can be seen that to first order $I(x)$ consists of current waves whose amplitudes decrease exponentially as they propagate along the antenna. The waves decay as $e^{-\beta_p |x|}$. By definition $\beta_p = \beta \sqrt{|P|}$, and thus the distribution of the current is determined by the value of P , the second diagonal element of the dielectric tensor given by (3b).

[61] To find a solution for (42) we first form the difference kernel, $K_d(x, x')$, by subtracting from both sides of (42) the integral:

$$H(h) = \frac{\mu_o \gamma}{8\pi^2} \int_0^{2\pi} \int_{-h}^h \frac{e^{-\beta_c \gamma R_a(h, x')}}{R_a(h, x')} I(x') dx' d\phi$$

[62] This action results in the expression

$$\frac{\mu_o \gamma}{4\pi} \int_{-h}^h K_d(x, x') I(x') dx' = i A_x(x) - H(h) \quad (43)$$

where $A_x(x)$ is defined in (37), and where

$$K_d(x, x') = \int_0^{2\pi} \left[\frac{e^{-\beta_c \gamma R_a(x, x')}}{R_a(x, x')} - \frac{e^{-\beta_c \gamma R_a(h, x')}}{R_a(h, x')} \right] \frac{d\phi}{2\pi}$$

[63] The left hand side of (43) is identically equal to 0 when $x = h$. This condition allows us to solve for the constant b_o in terms of V_o and $H(h)$. This leads to the final equation:

$$\begin{aligned} \frac{\mu_o \gamma}{4\pi} \int_{-h}^h K_d(x, x') I(x') dx' = \frac{\beta_p V_o}{2\omega} \frac{\sinh \beta_p (h - |x|)}{\cosh \beta_p h} \\ - H(h) \left(1 - \frac{\cosh \beta_p x}{\cosh \beta_p h} \right) \end{aligned} \quad (44)$$

[64] Equation (44) can be employed in general to find $I(x)$ using known methods [King *et al.*, 2002], once the values of β_c , β_p , γ , h , and ω are specified. Thus our primary goal of finding a method to determine $I(x)$ from first principles has been achieved.

2.2.1. Applicability of Triangular Current Distribution and Quasi-Static Model

[65] To fulfill our secondary goal, we can now make use of (44) to determine the conditions under which the current distribution can be approximated by a triangular current similar to (1b). It is clear that the right hand side of (44) can be reduced to a triangular distribution in x whenever $(\beta_p h)^2 \ll 1$. Since $\beta_p \simeq \omega_o / c$, where ω_o is the local angular plasma frequency, the current distribution will be approximately triangular only if $(\omega_o h / c)^2 \ll 1$. The parameter ω_o varies from $\sim 10^7$ /s at 600 km altitude to $\sim 2 \cdot 10^6$ /s at 6000 km altitude. This implies that the assumption of a triangular current distribution is appropriate only if h is substantially less than 30 m at 600 km altitude and substantially less than 150 m at 6000 km altitude. Furthermore the use of antennas exceeding these lengths is not productive, since the current moment cannot be significantly increased by increasing h .

[66] We can also use (44) to find the conditions under which quasi-static models, such as that of *Balmain* [1964], can reasonably be used to describe the electric field of the antenna. These models should be applicable as long as the wave attenuation along the antenna is small; ie, when $(\beta_c \gamma R_a)^2 \ll 1$ in (44). Since the maximum value of $R_a(x, x')$ is $R_a(x, x') = 2h$, the condition for the applicability of the quasi-static model can be written: $(2\beta_c h \gamma)^2 \ll 1$. For a practical constraint, let us set the limit as $(2\beta_c h \gamma)^2 \leq 0.1$. Given this constraint, we can define the maximum allowable value of h as: $h_m \simeq 0.15 / \beta_c \gamma$. From the definitions of β_c and γ it can be shown that $\beta_c \gamma = \sqrt{2} \beta_p / \sqrt{1 - f_{lhr}^2 / f^2}$, an expression essentially independent of f when $f^2 \gg f_{lhr}^2$. Making use of the definition of the product $\beta_c \gamma$, the constraint can be written: $h_m \simeq 0.1 \sqrt{1 - f_{lhr}^2 / f^2} / \beta_p$. If we now assume that $f^2 \gg f_{lhr}^2$, then at 600 km altitude with $\omega_o \simeq 10^7$ /s we

find that $h_m \simeq 3$ m, while at 6000 km altitude with $\omega_o \simeq 2 \cdot 10^6$ /s, $h_m \simeq 15$ m. In the event that $f \simeq f_{lhr}$, the value of h_m will be even smaller. For example If we assume that $f = 1.1f_{lhr}$, then at 6000 km altitude $h_m \simeq 1$ m. Thus the quasi-static model would appear to be applicable to the perpendicular antenna only over a much smaller range of antenna lengths than was the case for the parallel antenna.

2.2.2. Antenna Input Impedance

[67] If (44) is valid, it should yield the correct input impedance of the antenna. To test this relationship, we can compare the predictions of (44) with the results of *Balmain* [1964], *Wang and Bell* [1969], and *Wang* [1970]. In these studies a triangular current distribution was assumed, implying $(\beta_p h)^2 \ll 1$. To proceed we assume that $(\beta_p h)^2 \ll 1$ and approximate the hyperbolic functions in (44) by the relations: $\sinh \beta_p(h - |x|) \simeq \beta_p(h - |x|)$, $\cosh \beta_p h \simeq 1$, and $\cosh \beta_p x \simeq 1$. This action leads to the expression

$$\frac{\mu_o \gamma}{4\pi} \int_{-h}^h K_d(x, x') I(x') dx' = \frac{\beta_p^2 V_o}{2\omega} (h - |x|) \quad (45)$$

[68] Again, following the procedure outlined by *King et al.* [2002, chapter 2], it can be shown that the first-order solution to (45) is given by the expression

$$I(x) = \frac{2\pi V_o}{Z_o \gamma \Psi} \beta_p (h - |x|) \quad (46a)$$

where

$$\Psi = \int_{-h}^h \left(1 - \frac{|x'|}{h}\right) K_d(0, x') dx' \quad (46b)$$

[69] The maximum value of the argument of the exponential functions in (46b) is $\beta_c \gamma h$. Equation (46b) can be evaluated analytically in both the quasi-static limit, for which $(\beta_c \gamma h)^2 \ll 1$, and in the electromagnetic limit, for which $(\beta_c \gamma h)^2 \gg 1$. If $(\beta_c \gamma h)^2 \gg 1$, we can closely approximate the exponentials by the two term Taylor series: $e^{-x} \simeq 1 - x$. With this action, the integrations in (46b) become straightforward and it is found that

$$\Psi_{qs} = 2 \left[\log\left(\frac{2h}{a}\right) - 0.8 \right] - i \tan^{-1} \left(\frac{1}{\gamma} \right) \quad (46c)$$

where it is assumed that $\log\left(\frac{2h}{a}\right) \gg 1$. Equation (46c) represents the predictions of our model in the quasi-static

limit. On the other hand, for $(\beta_c \gamma h)^2 \gg 1$ we have the electromagnetic limit for which (46b) becomes

$$\Psi_{em} \simeq 2 \left[\log\left(\frac{2h}{a}\right) - \log(\beta_c \gamma h) \right] - i \tan^{-1} \left(\frac{1}{\gamma} \right) \quad (46d)$$

where it is assumed that $\log\left(\frac{2h}{a}\right) \gg 1$, and $(\beta_c \gamma a)^2 \ll 1$.

[70] From the preceding we can find the antenna input impedance for the perpendicular dipole:

$$Z_{in} = V_o / I(x=0) = R_{r\perp} + iX_{\perp}$$

where

$$R_{r\perp qs} = \frac{Z_o}{\pi h \beta \sqrt{|P|S}} \left[\log\left(\frac{2h}{a}\right) - 0.8 \right] \quad (47a)$$

$$R_{r\perp em} \simeq \frac{Z_o}{\pi h \beta \sqrt{|P|S}} \left[\log\left(\frac{2h}{a}\right) - \log(\beta_c \gamma h) \right] \quad (47b)$$

$$X_{\perp} = - \frac{Z_o}{\pi h \beta \sqrt{|P|S}} \arctan\left(\frac{1}{\gamma}\right) \quad (47c)$$

where (47a) applies when $(\beta_c \gamma h)^2 \ll 1$, (47b) applies when $(\beta_c \gamma h)^2 \gg 1$, and where terms of order $1/\gamma^2$ in the denominators of (47) have been neglected.

[71] Equations (47) have been developed under the assumption that the refractive index $n(\psi)$ extends to large values for wave normals near the resonance cone, as depicted in Figure 1. This situation occurs for $f \geq f_{lhr}$. However $n(\psi)$ also extends to large values as $\psi \rightarrow \pi/2$ for f slightly below f_{lhr} where the condition $(\beta_c \gamma h)^2 \gg 1$ still applies. In this frequency range we can still apply (44), but we need to take into account the fact that the dielectric constant S in (3b) is negative in this frequency range, and thus the ratio P/S is now positive. Equation (44) can be readily modified for the new frequency range by substituting the product $i\gamma$ for each γ that appears in (44). This change results in the expression

$$\begin{aligned} \frac{\mu_o i \gamma}{4\pi} \int_{-h}^h K_{d2}(x, x') I(x') dx' &= \frac{\beta_p V_o}{2\omega} \frac{\sinh \beta_p (h - |x|)}{\cosh \beta_p h} \\ &- H_2(h) \left(1 - \frac{\cosh \beta_p x}{\cosh \beta_p h} \right) \end{aligned} \quad (48)$$

where

$$K_{d2}(x, x') = \int_0^{2\pi} \left[\frac{e^{-i\beta_c \gamma R_{a2}(x, x')}}{R_{a2}(x, x')} - \frac{e^{-i\beta_c \gamma R_{a2}(h, x')}}{R_{a2}(h, x')} \right] \frac{d\phi}{2\pi}$$

$$R_{a2}(x, x') = \left[(x - x')^2 + (a \sin\phi)^2 + (a \cos\phi)^2 / \gamma^2 \right]^{1/2}$$

$$H_2(h) = \frac{\mu_0 i \gamma}{8\pi^2} \int_0^{2\pi} \int_{-h}^h \frac{e^{-i\beta_c \gamma R_a(h, x')}}{R_a(h, x')} I(x') dx' d\phi$$

where $\gamma = \sqrt{P/S}$.

[72] It can be seen from (48) that $R_{a2}(x, x')$ is always real and positive, and that there are no singularities associated with the kernel $K_{d2}(x, x')$.

[73] We wish to find the antenna input impedance from (48) for the case of a triangular current distribution. We assume that $(\beta_p h)^2 \ll 1$ on the right hand side of (48) and proceed exactly as described in the text following (44), first calculating the function Ψ and then the antenna input impedance Z_{in} . We calculate Ψ in both the quasi-static limit, for which $(\beta_c \gamma h)^2 \ll 1$, and the electromagnetic limit, for which $(\beta_c \gamma h)^2 \gg 1$. Following this procedure we arrive at the results

$$R_{r\perp qs} = 0 \quad (49a)$$

$$X_{\perp qs} = \frac{Z_o}{\pi \beta h \sqrt{PS}} \left[\log\left(\frac{2h}{a}\right) - 0.8 \right] \quad (49b)$$

$$R_{r\perp em} = \frac{Z_o}{2\beta h \sqrt{PS}} \quad (49c)$$

$$X_{\perp em} = \frac{Z_o}{\pi \beta h \sqrt{PS}} \left[\log\left(\frac{2h}{a}\right) - \log(\beta_c \gamma h) \right] \quad (49d)$$

where it is assumed in (49a) and (49b) that $(\beta_c \gamma h)^2 \ll 1$, and it is assumed in (49c) and (49d) that $(\beta_c \gamma h)^2 \gg 1$ and $(\beta_c \gamma a)^2 \ll 1$. Equations (49a) and (49b) essentially represent the quasi-static solution for the input impedance. It can be seen from (49a) that the quasi-static model fails to predict a finite value for the radiation resistance when $f < f_{thr}$. On the other hand, when $f > f_{thr}$ and $(\beta_c \gamma h)^2 \gg 1$, our electromagnetic model predicts a finite value for $R_{r\perp}$ and a smaller value for X_{\perp} , as given in (49c) and (49d).

[74] As discussed below, the values of $R_{r\perp}$ and X_{\perp} in (47) and the value of $R_{r\perp}$ in (49a) are very close to the leading terms of the values calculated in the full-wave solutions of *Wang and Bell* [1969] and *Wang* [1970]. This gives confidence that the integral equation (44)

derived in the short wavelength limit contains the essential physics of the radiation process.

3. Comparison With Past Work

[75] In the event that the antenna current, $J_a(r)$, is a known quantity, then (2) and (3) can be solved using three dimensional Fourier spatial transforms in order to find the E and B fields produced by the antenna current as well as the radiation resistance and reactance of the antenna. The solutions are expressed as integrals over the Fourier k space. Such solutions have been found in the past [*Wang and Bell*, 1969; *Wang*, 1970] for the case in which the antenna is assumed to have a triangular current distribution similar to (1b).

[76] Given a triangular current distribution, the radiation resistance and reactance of the antenna is found to be a function of the angle between the dipole antenna axis and the local direction of the Earth's magnetic field B_o . The leading terms of the predicted radiation resistance and reactance for the two cases in which the antenna axis is either parallel to B_o or perpendicular to B_o are given by the expressions [*Wang and Bell*, 1969; *Wang*, 1970]

$$R_{r\parallel} = \frac{Z_o}{2h\beta S} \quad (50a)$$

$$X_{\parallel} = -\frac{Z_o}{\pi h\beta S} \left[\log\left(\frac{2h}{\gamma a}\right) - 1 \right] \quad (50b)$$

$$R_{r\perp} = \frac{Z_o}{\pi h\beta \sqrt{|P|S}} \left[\log\left(\frac{2h}{a}\right) - 1 \right] \quad (50c)$$

$$R_{r\perp} = \frac{Z_o}{\pi h\beta \sqrt{|P|S}} \left[\log\left(\frac{2h}{a}\right) - \log(\beta_c \gamma h / \sqrt{2}) \right] \quad (50d)$$

$$X_{\perp} = -\frac{Z_o}{\pi h\beta \sqrt{|P|S}} \arctan\left(\frac{1}{\gamma}\right) \quad (50e)$$

$$R_{r\perp} = \frac{Z_o}{2\beta h \sqrt{PS}} \quad (50f)$$

where (50a) and (50b) apply when $(h\beta_c)^2 \ll 1$, (50c) applies when $(\beta_c \gamma h)^2 \ll 1$, (50d) is obtained from equation (B3) of *Wang and Bell* [1969] in the limit $(\beta_c \gamma h)^2 \gg 1$ with $f > f_{thr}$, and (50f) is obtained from equation (B1) of *Wang and Bell* [1969] in the limit $(\beta_c \gamma h)^2 \gg 1$ with $f < f_{thr}$.

[77] It can be seen that (50a) agrees well with (32a), and that (50b) agrees well with (32b). Also (50c) agrees well with (47a), (50d) agrees well with (47b), and (50e) agrees well with (47c). Furthermore (50a), (50b), (50c), and (50e) agree well with results of the quasi-static model [Balmain, 1964]. It is noteworthy that the quasi-static model predicts $R_{r\perp} = 0$ for $f < f_{thr}$. This comes about because in the quasi-static model $\beta_c \gamma h \equiv 0$, and when $f < f_{thr}$, the kernel $K_{d2}(x, x')$ is always real, and the input reactance is then always purely imaginary. The close agreement between the values of the dipole input impedance calculated from the full set of Maxwell's equations and from our integral equations, (28) and (44), gives confidence that the integral equations contain the essential physics of the electromagnetic wave radiation process.

4. Summary and Discussion

[78] Working in the short wavelength limit, we have constructed two integral equations from first principles from which the current distribution can be found for an electric dipole antenna in a magnetized plasma oriented either parallel or perpendicular to the ambient magnetic field B_o . We have applied these equations to the case in which the dipole antenna is located at high altitudes within the Earth's radiation belts. We have found that although for a fixed VLF frequency the antenna will radiate waves with wavelengths ranging from kilometers to meters, the antenna current distribution depends primarily upon the diagonal elements of the dielectric tensor. In the case in which the antenna is parallel to B_o , the diagonal dielectric constant S determines the wavelength of the current waves along the antenna. In the case in which the antenna is perpendicular to B_o , the diagonal dielectric constant P determines the decay constant of the evanescent current waves along the antenna.

[79] We have found that the assumption of a triangular current distribution such as (1b) for the parallel antenna appears reasonable for all antenna lengths for which the short wavelength limit applies. On the other hand, for the perpendicular antenna it was found that the triangular current assumption was reasonable only when the total antenna length was substantially less than 60 m at 600 km altitude and substantially less than 300 m at 6000 km altitude. Furthermore the use of antennas with total lengths exceeding these limits was not constructive since the dipole moment of the antenna cannot be significantly increased by increasing the antenna length.

[80] Using the integral equations, we have calculated the input impedance of the antenna parallel to B_o for the case in which the dipole length was significantly smaller than the wavelength of the current waves along the antenna and found that it agreed closely with the leading

terms of the input impedance for a triangular current distribution calculated from the full set of Maxwell's equations as expressed in (50a) and (50b). We have also calculated the input impedance of the antenna perpendicular to B_o for the case in which the dipole length was significantly smaller than the reciprocal of the spatial decay constant of the evanescent current waves along the antenna and found that it agreed closely with the leading terms of the input impedance for a triangular current distribution calculated from the full set of Maxwell's equations, as expressed in (50c), (50d), (50e), and (50f). This close correspondence of results gives confidence that the integral equations derived in the short wavelength limit contain the essential physics of the radiation process.

[81] Comparing our results for the antenna input impedance with the predictions of the quasi-static model [Balmain, 1964], it was found that in the case of the antenna parallel to B_o the two methods were in good agreement for all antenna lengths for which the integral equation of (28c) was applicable. In the case of the antenna perpendicular to B_o , our results suggest that the quasi-static model can reasonably describe the antenna input impedance only when the total antenna length is less than 30 m at 6000 km altitude and less than 6 m at 600 km altitude.

[82] The extension of our method to the case in which the dipole is oriented at some angle with respect to B_o other than 0° or 90° is straightforward, but the integral equations for the antenna current are much more complicated. We were able to obtain a relatively simple integral equation for the dipole current because of our choice of $C_1 = S$ for the parallel dipole, and our choice of $C_1 = 2(P - S)$ for the perpendicular dipole. The current of a dipole oriented at some intermediate angle with respect to B_o can be decomposed into currents along both the parallel and perpendicular directions. However, although there are two current components, we can choose only a single value for C_1 . Since $|P - S| \gg |S|$, no single choice of C_1 appears to lead to Green's functions as simple as (19) or (38) for both A_z and A_x .

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