NON-LINEAR GYRORESONANT INTERACTIONS OF ENERGETIC PARTICLES 
AND COHERENT VLF WAVES IN THE MAGNETOSPHERE

by

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Radioscience Laboratory
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This research is a Ph. D. dissertation submitted to Stanford University. It is dedicated to my mother and father

HAYRIYE and MUSTAFA INAN

and to my wife

ELIF INAN
ABSTRACT

The study of wave particle interactions in the earth's magnetosphere has advanced markedly in the last decade. Understanding of these interactions is important because of their possible impact on the ionosphere in general and VLF and ULF communications in particular, their control of the radiation belt particles and the role played by the waves in diagnostics of the magnetosphere. One important class of wave particle interactions is the gyroresonant interaction of coherent VLF whistler mode waves and energetic particles. The waves involved can be natural whistlers or discrete emissions or signals injected into the magnetosphere from VLF ground transmitters, such as the Stanford University transmitter at Siple, Antarctica, and large scale power grids. These coherent waves interact in the cyclotron-resonance mode with the energetic particles trapped in the radiation belts. As a result the waves grow or decay in amplitude and the particles are perturbed in pitch angle and energy. The perturbations in pitch angle are of special importance in that they result in precipitation of particles into the ionosphere. Until recently this effect has only been studied using linear theory and analytical techniques. In the present work, a computer simulation study of this interaction is made with special emphasis on computing the wave's effect on the particles. With this approach it is possible to obtain a full nonlinear solution of the equations of motion in an inhomogeneous medium. The nonlinear results are compared with those of the linear theory and a convenient criterion is presented for determining when a complete nonlinear solution is required. It is found, for example, that in the case of equatorial scattering by a 5 kHz CW pulse near L = 4
linear theory begins to break down when the wave amplitude exceeds 3 m/s. We represent a full distribution of energetic particles by 40-50,000 test particles, distributed appropriately in phase space. By computing the complete trajectories of all these particles the perturbation of the full distribution is estimated. The wave induced precipitated flux is computed and it is shown that significant particle fluxes (order of $10^{-1}$ ergs/cm²-sec) can be precipitated into the atmosphere by waves of moderate intensity (order of 10 m/s). These fluxes produce significant perturbations in the nighttime ionosphere. Both the incoming fluxes and the ionospheric perturbations appear to be measurable by presently available instruments.
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<td>$B_o$</td>
<td>geomagnetic field</td>
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<tr>
<td>$B_w$</td>
<td>wave magnetic field</td>
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<tr>
<td>$e$</td>
<td>electronic charge, $1.602 \times 10^{-19}$ coulombs</td>
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<tr>
<td>$f$</td>
<td>wave frequency, $\omega/2\pi$</td>
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<td>$f_H$</td>
<td>electron gyrofrequency, $\omega_H/2\pi$</td>
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<td>$f_P$</td>
<td>electron plasma frequency, $\omega_p/2\pi$</td>
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<tr>
<td>$f(v,\alpha)$</td>
<td>energetic particle distribution functions</td>
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<tr>
<td>$f(v_\perp,v_\parallel)$</td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>wave number</td>
</tr>
<tr>
<td>$K_E$</td>
<td>kinetic energy, $\frac{1}{2}mv^2$</td>
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<td>$L$</td>
<td>equatorial geocentric distance (in earth radii) of the magnetic field line. Defined on p. 15, Eq. (2.3).</td>
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<tr>
<td>$L_I$</td>
<td>interaction length</td>
</tr>
<tr>
<td>$m$</td>
<td>mass of an electron, $9.109 \times 10^{-31}$ kg.</td>
</tr>
<tr>
<td>$n_{eq}$</td>
<td>equatorial cold plasma density</td>
</tr>
<tr>
<td>$n$</td>
<td>wave refractive index</td>
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<tr>
<td>$N_T$</td>
<td>total number density of precipitated electrons</td>
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<td>$N_e$</td>
<td>number density of isotropically distributed energetic electrons in the 1-2 keV energy range.</td>
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<tr>
<td>$Q$</td>
<td>precipitated energy deposition rate</td>
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<tr>
<td>$R$</td>
<td>geocentric distance</td>
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<td>$R_o$</td>
<td>radius of the earth, 6370 km.</td>
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<td>$v_{p,g}$</td>
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<td>$v$</td>
<td>energetic electron velocity</td>
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LIST OF PRINCIPAL SYMBOLS (Cont.)

\( v_{\parallel, \perp} \) = components of the energetic particle velocity parallel and perpendicular to the static magnetic field.

\( v_R \) = local cyclotron-resonance velocity, \( \frac{\omega_{ci} - \omega}{k} \).

\( v_k \) = 1.9 \times 10^7 \text{ m/sec}, velocity corresponding to 1 keV energy, \( \frac{1}{2} m v^2 = 1 \text{ keV} \).

\( z \) = distance along the field line.

\( \alpha \) = particle pitch angle, \( \tan^{-1} \left( \frac{v}{v_{\parallel}} \right) \).

\( \alpha_{lc} \) = half angle of the loss cone. Defined on p. 23.

\( \lambda \) = geomagnetic latitude.

\( \theta \) = wave normal angle, \( \frac{1}{2} (k, B_0) \).

\( \tau_B \) = bounce period of energetic particles.

\( \delta \) = wave polarization.

\( \rho \) = ratio of wave and inhomogeneity forces. Defined on p. 91.

\( \phi \) = phase of the energetic electron with respect to the wave, \( \frac{1}{2} (v_{\parallel}, -B_w) \).

\( \phi_u \) = unperturbed phase, i.e. \( \phi \) for \( B_w = 0 \).

\( \eta \) = ratio of the percentage pitch angle and energy changes. Defined on p. 53.

\( \phi \) = differential energy spectrum (number of electrons per unit area per unit time per unit solid angle per unit energy).

\( \phi_{\perp} \) = differential energy spectrum of \( \sim 1 \text{ keV} \) electrons with \( \alpha = 90^\circ \).

\( \Delta(\cdot) \) = change in the quantity \( (\cdot) \).

\( (\cdot)_{eq} \) = equatorial value of the quantity \( (\cdot) \).
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I. INTRODUCTION

The subject of this research is the nonlinear cyclotron resonance wave-particle interaction in the magnetosphere, with particular emphasis on the wave-induced perturbations of the energetic particles. In this section we give a brief introduction to the topic and describe the present work and the specific contributions.

A. THE EARTH'S INNER MAGNETOSPHERE

The immediate environment of the earth consists of (i) the neutral atmosphere, extending up to 60 km above the earth's surface, (ii) the ionosphere, a region of highly ionized gas (heavy ions, protons and electrons) as well as neutral particles extending up to about 1000 km altitude, and (iii) the magnetosphere consisting mainly of protons and electrons and extending up to a distance of ~100,000 km from the earth. Although the boundary between the atmosphere and the ionosphere is rather abrupt, such is not the case for the ionosphere-magnetosphere boundary. The magnetosphere could be viewed as a smooth continuation of the ionosphere and has often been referred to as the 'outer ionosphere.'

The structure and the physical processes of the magnetosphere are controlled mainly by the earth's magnetic field and the energy from the sun. This energy arrives at the earth in different forms and is deposited at different locations. Some is introduced at the outer boundary by the solar wind, a hot ($10^5$ K) solar gas, mainly protons and electrons traveling outward from the sun at speeds of ~500 km/s. Containing 5-10 charged particles per cm$^3$ at 1 A.U., the solar wind is prevented from reaching the earth itself by the effects of the earth's magnetic field. A bow-shock is formed at -12 R$_o$, with the solar wind dragging magnetic lines of
force around the earth in the anti-solar direction, forming a large tail containing plasma and magnetic field. A sketch of the magnetosphere is shown in Fig. 1.1. For a thorough discussion of solar wind and magnetosphere interaction the reader is referred to Akasofu and Chapman [1972].

The region of interest for this report is the inner magnetosphere. This is the region within about 7 earth radii of the earth. A good approximation to the earth's magnetic field in this region is that of a centered dipole inclined with respect to the rotation axis by about 11°. A sketch of the inner magnetosphere and the dipole field lines is shown in Fig. 1.2.

The inner magnetosphere is populated by background 'cold' plasma, mainly electrons and protons in the 0.1 eV to 1 eV energy range. Often the inner magnetosphere exhibits an abrupt boundary, called the plasmapause, at which the electron concentration drops by roughly two orders of magnitude [Carpenter, 1963]. Its geocentric distance is highly variable ranging from 2 earth radii after the onset of a major substorm to as much as 7 earth radii after a long (several days) period of quieting. The region inside the plasmapause is called the plasmasphere [Carpenter and Park, 1973]. The thermal or 'cold' plasma is created through ionization by solar ultra-violet radiation in the daytime F region of the ionosphere, and flows upward along field lines into the plasmasphere. At night, ionization stored in the plasmasphere flows downward to help maintain the nighttime F region [Park, 1970]. The cold plasma in the plasmasphere is commonly assumed to be a neutral isothermal mixture of electrons and positive ions (mainly H⁺, but some He⁺ and O⁺) which is in diffusive equilibrium along the magnetic field lines under the effect of
Figure 1.1 The General Outline of the Magnetosphere
the earth's gravitational and centrifugal forces [Angerami and Thomas, 1964]. The cold plasma densities in the plasmasphere range from $10^4$ particles/cc at 1000 km altitude to -300-400 particles/cc at the inner edge of the plasmasphere.

In addition to the cold plasma, the inner magnetosphere is also populated by energetic (hot) particles which constitute the earth's radiation belts. These particles, mainly protons and electrons with energies from 10 keV up to 100 MeV, are magnetically trapped in the earth's field. They execute a helical gyro motion around the field lines and bounce back and forth between conjugate hemispheres. Typical energetic particle trajectories are shown in Fig. 1.2.

The study of the radiation belts has been increasingly important and relevant for almost all areas of magnetospheric physics in the last two decades [Hess, 1968; Roederer, 1970; Akasofu and Chapman, 1972].

Although the source and loss processes of these energetic particles are not fully understood, one possibility is that the particles of the solar wind enter the magnetosphere through the tail, are accelerated along the magnetic field lines and enter the plasmasphere at the sunward side after arriving there through the mechanism of cross-field drifts. In addition to being trapped along the field lines the energetic particles drift in the azimuthal direction due to both $E \times B$ and gradient $|B|$ drifts.

An important loss process for the radiation belt particles is wave-particle interactions. The magnetosphere is extremely rich in the kinds of waves that it supports. As an excellent and unique example of a dense, 'infinite' magnetoplasma, it supports wave modes in a broad range
of frequencies from 1 Hz up to 100 MHz. Almost all of these modes exhibit fundamental wave properties like dispersion and anisotropy. Some of these modes (e.g., the whistler mode) can resonate with the energetic particles due to their low wave phase velocities [Brice, 1964]. During these wave-particle resonant interactions the waves are amplified or damped through various plasma instability mechanisms and the particles are scattered in energy and pitch angle, the latter being the angle of inclination of the particle velocity vector with respect to the magnetic field direction. As a result of these perturbations some particles are precipitated out of the radiation belts and into the atmosphere. These precipitated particles produce secondary ionization, emit x-rays and in general produce significant perturbations in the lower ionosphere.

Wave particle interactions can be roughly classified as being either incoherent or coherent. The former involves incoherent wide band electromagnetic waves such as ELF-VLF plasmaspheric hiss. In these interactions the forces exerted by the wave on the particle are uncorrelated and the particles execute a random walk in velocity space. Coherent interactions involve narrowband waves such as VLF signals from ground transmitters, natural whistlers, triggered emissions and signals induced by large scale power grids. During such coherent interactions the wave induced forces on the particle are cumulative in nature. The particles can therefore be phase locked with the coherent wave and execute well defined motions during which they suffer significant perturbations in energy and momentum. Much work over the past decade has been devoted to the study of incoherent wave particle interactions. Relatively little effort has been made to study coherent interactions.
The purpose of the research presented in this report is to determine the nature of particle precipitation caused by the cyclotron resonance interaction between a coherent whistler-mode signal and energetic particles in the 0.5 keV to 200 keV range. An important characteristic of the whistler mode is that \( f < f_H \), where \( f \) is the wave frequency and \( f_H \) is the electron gyrofrequency. The wave polarization is elliptical in general and right-hand circular for the case of propagation along the static magnetic field, with the wave vectors rotating in the same sense as a gyrating electron for the propagating mode. Furthermore the wave phase velocity is of the order of 0.01c-0.1c. Therefore these 'slow' waves can achieve both longitudinal and cyclotron resonances with relatively low energy particles [Brice, 1964].

The cyclotron (or gyro) resonance interaction is an interaction in which the doppler-shifted wave frequency seen by the electrons is equal to the electron gyrofrequency. In that case the particles experience an approximately stationary wave field for an extended period of time; significant cumulative interactions can occur resulting in exchange of energy between the wave and the particles through the wave's electric field. The pitch angle, i.e. the direction of the particle velocity, also changes due to the transfer of the particle's parallel (along static magnetic field) momentum to perpendicular momentum (or vice-versa) under the influence of the wave's magnetic field. When the particle pitch angle is lowered below a value called the 'loss cone' pitch angle the particle reaches the atmosphere and is precipitated. It is this wave induced pitch angle scattering by coherent VLF whistler mode waves that is the subject of this report.
In the following we give a brief review of previous work done in the field of pitch angle scattering due to wave-particle interactions.

B. REVIEW OF PREVIOUS WORK

There has been considerable work done on the pitch angle scattering of radiation belt particles by electromagnetic waves [Dungey, 1963, 1964; Kennel and Engelmann, 1966; Kennel and Petschek, 1966; Roberts, 1966, 1968, 1969; Gendrin, 1968; Kennel, 1969; Lyons et al., 1971, 1972; Ashour-Abdalla, 1972; Schulz and Lanzerotti, 1973; Lyons, 1973, 1974a,b]. Most of this work, however, has addressed the problem of scattering by wideband, incoherent whistler mode turbulence. The idea has been that the trapped particle population interacts through cyclotron resonance with electromagnetic disturbances along its orbit and is subjected to a series of scatterings that are random in both direction and size. Hence the individual particles of the population undergo a random walk in pitch angle, and diffusion in equatorial pitch angle space results. This diffusion can then be studied by calculating the incoherent diffusion coefficients and solving a Fokker-Planck equation [Roberts, 1966].

This approach is well justified for studying the scattering due to interaction with certain kinds of magnetospheric signals, for example, auroral VLF hiss or ELF plasmaspheric hiss, since such waves are indeed wide band and highly incoherent [Muzzio, 1971; Gurnett and Frank, 1972; Thorne et al., 1973; Laaspere and Hoffman, 1976].

The physics in our case is fundamentally different because it involves highly coherent, narrowband whistler mode waves. When a particle population encounters such coherent waves, the series of scatterings experienced by the particles are not random in direction or size. The in-
dividual particles of the population can be phase locked with the co-
herent signal for distances of many hundred wavelengths and undergo large
pitch angle changes in a single encounter with the wave. It is there-
fore incorrect to assume that the particles execute a random walk in
pitch angle during the course of one bounce period when interacting with 
coherent waves.

The study of wave–particle interactions with coherent waves is very
important. Examples of such highly coherent magnetospheric signals are
natural whistlers [Helliwell, 1965], triggered VLF emissions [Stiles and
Helliwell, 1975], signals that are injected into the magnetosphere by
VLF ground transmitters [Helliwell and Katsufrakis, 1974] and large scale
power grids [Helliwell et al., 1975; Park, 1976] or signals from satellite
borne VLF transmitters, such as that planned for the AMPS mission.

During the past three years VLF wave injection experiments have been
carried out using the Stanford University variable frequency VLF trans-
mitter at Siple Station in the Antarctic [Helliwell and Katsufrakis,
1974]. The Siple wave injection experiment is an active experiment de-
signed to study coherent VLF wave–particle interactions in the magneto-
sphere. One goal of the experiment is to learn how to control the ener-
ggetic particles by the injected waves.

Once control is established, the energetic particles can be used
as tools to study other important processes. For example, the control
of energetic particle precipitation would permit controlled studies of
x-ray, ionization and radiation emission processes in the ionosphere.
The generation and study of precipitation induced modifications in the
D-region [Helliwell et al., 1973] would also be facilitated. Further-

- 9 -
more, modulation of precipitation flux might provide a means to produce Pc-1 ULF waves [Bell, 1976] on a controlled basis. Reducing the particle population in the radiation belts with appropriate transmissions is another possible future application. Although ground transmitters illuminate a relatively large region of the magnetosphere [Inan et al., 1977a], satellite transmitters may be necessary for applications requiring high wave amplitudes.

Theoretical studies of the coherent cyclotron resonance wave-particle interaction have concentrated on wave growth and generation due to the phase bunching of the energetic particles [Brice, 1964; Bell and Buneman, 1964; Helliwell, 1967, 1970; Dysthe, 1971; Nunn, 1971, 1974; Palmadesso and Schmidt, 1971, 1972; Matsumoto, 1972; Brinca, 1972; Bud'ko et al., 1972; Helliwell and Crystal, 1973; Karpman et al., 1974a,b].

Pitch angle scattering induced by coherent waves have been considered by only a few authors [Das, 1971; Ashour-Abdalla, 1972]. However a linear theory was employed in each of these studies. Although it is generally accepted that linear theory applies for small wave amplitudes, none of the authors who have used this theory has given quantitative justification for the assumptions. The main advantage of the theory is that it considerably simplifies the analysis.

C. OUTLINE OF THE REPORT

This report describes a computer simulation approach to the study of the cyclotron resonance wave particle interaction between coherent VLF whistler mode signals and energetic electrons in the magnetosphere. In particular, we consider the wave-induced pitch angle scattering of the particles. One important result of this scattering is precipitation
of particles into the atmosphere.

We have employed a Lagrangian formulation involving a test particle simulation of the nonlinear equations of motion. In this approach, the effect of a wave signal on a particle population is calculated by simulating the interaction for a sufficient number of test particles. Although the purpose of this simulation is to compute the wave's effect on the particles, our results are also directly applicable to the problem of wave growth and generation through phase bunching [Helliwell, 1967; 1970], since we compute the full phase motion of all test particles in order to obtain the total scattering.

Since we use a test particle approach, our calculations do not include the effects of the electromagnetic fields generated by the perturbed energetic particles. In effect we assume that the currents stimulated in the energetic particle population do not lead to significant damping or amplification of the wave near the magnetic equatorial plane. Experimental results indicate that this assumption holds true a good deal of the time in the magnetosphere. A discussion of this point is included in Chapter 6.

In this report we have considered only the case of a monochromatic whistler mode signal. However the computer program is quite general and is capable of dealing with the case of single pulses or a train of pulses. The case of a single frequency is not as limited as it might seem, since there is some reason to believe that the interaction with a wave with linearly increasing or decreasing frequency is quantitatively not much different [Helliwell, 1970]. We also limit ourselves to wave propagation strictly along the static magnetic field lines with \( k \) parallel to \( \vec{B}_0 \).
We assume that the wave is not affected by the energetic particles and has a uniform amplitude over the field line near the equatorial plane. The energetic particles also do not interact with each other. This assumption allows the computation of the particle trajectories sequentially instead of in parallel.

The computer simulation is versatile and can be used for a variety of magnetospheric conditions. In this report we give results for a 5 kHz wave signal propagating along the dipole field line which has an equatorial geocentric distance of about 4 Earth radii.

The organization of the report is as follows. Chapter 2 provides the basic physics and derives the equations of motion. The trapped particles and the coherent VLF waves that exist in the magnetosphere and the principles of the gyroresonant interaction through which these two affect each other are described there.

In Chapter 3 we describe the computer simulation of the interaction. The computer program is given in an Appendix.

Chapter 4 treats the nonlinear trajectories of single resonant particles and sheets of resonant particles distributed in cyclotron phase. The purposes of this chapter are (i) to understand the physics of the interaction and the dependence of total scattering on different wave, medium and particle parameters, and (ii) to define the limitations of linear theory.

In Chapter 5 we give an application of the simulation to a sample computation of the precipitated flux using a full distribution of particles. We present the results in units that are comparable with measurements.
In the last chapter, conclusions, discussion and suggestions for future work are given.

D. CONTRIBUTIONS OF THE PRESENT WORK

1) We have for the first time computed the precipitated fluxes due to interaction of a full distribution of energetic particles and coherent VLF waves. Our results show that significant energy fluxes \((10^{-3} \text{ergs/cm}^2\text{-sec})\) can be deposited by moderate intensity \((\sim10 \text{ mV})\) coherent waves. These fluxes produce significant perturbations in the lower ionosphere. Both the incoming fluxes and the perturbations produced by them are well within the resolution of current instruments.

2) Our results show that nonlinear effects are significant for wave amplitudes as low as \(3 \text{ mV}\). We have also formulated a convenient quantitative criterion for determining the applicability of linear theory under given conditions.

3) The complete nonlinear equations of motion for particles that cyclotron resonate with a whistler-mode wave in an inhomogeneous magnetosphere are simulated on a computer. It is shown that the inhomogeneity of the medium can be incorporated with little additional computer time, by employing spatial instead of temporal steps in the simulation.

4) The straightforward computer simulation of the wave particle interaction has led to a clear presentation of several different aspects of the physics of the interaction such as wave trapping and relative effects of the wave and inhomogeneity forces. We have shown the dependence of the particle perturbations on a number of wave, medium and particle parameters.
5) Our work has provided a basis for future studies which may be undertaken with relatively minor modifications of the present computer simulation. Some of these are described in a section of Chapter 6 on "suggestions for future work."
II. BASIC PHYSICS AND EQUATIONS OF MOTION

A. THE MODEL MAGNETOSPHERE

The earth's magnetic field in the inner magnetosphere can be closely approximated by a centered dipole inclined with respect to the rotation axis by about 11°. The magnetic field strength in that case is given by

\[ B_0 = 0.312 \times 10^{-4} \left( \frac{R_o}{R} \right)^3 \left( 1 + 3 \sin^2 \lambda \right)^{1/2} \frac{\omega b}{m^2} \]  

(2.1)

where \( \lambda \) is the geomagnetic latitude, \( R \) is the geocentric distance and \( R_o \) is the mean radius of the earth, 6370 km. Figure 2.1 illustrates the dipole geometry and the symbols used.

A dipole field line is described by the relation

\[ \frac{R}{2 \cos \lambda} = \frac{R_o}{2 \cos \lambda_o} \]  

(2.2)

\[ \psi = \psi_o \]

where the subscript "o" refers to the earth's surface and \( \psi \) is the geomagnetic longitude. The field lines in any meridional plane can be uniquely identified by a parameter \( L \) defined as

\[ L = \frac{R_{eq}}{R_o} = \frac{1}{\cos^2 \lambda_o} \]  

(2.3)

where \( R_{eq} \) is the geocentric distance of the field line at the equator. For example the \( L = 4 \) field line crosses the geomagnetic equator at a geocentric distance of 4 earth radii and enters the earth at geomagnetic
latitudes \( \pm 60^\circ \).

The cold plasma in the inner magnetosphere can be approximated by a diffusive equilibrium model in which the neutral isothermal mixture of electrons and positive ions (\( H^+ \), \( \text{He}^+ \) and \( O^+ \)) is in diffusive equilibrium along the magnetic field lines, under the influence of the earth's gravitational and centrifugal forces [Angerami and Thomas, 1964]. The electron density in this case is given by [Park, 1972]
\[ n_e = n_1 \left[ \sum c_i \exp \left( \frac{z_G}{H_i} \right) \right]^{1/2} \]

\[ z_G = R_1 - \frac{R_1^2}{R} - \frac{\Omega}{2g_1} \left( R^2 \cos^2 \lambda - R_1^2 \cos^2 \lambda_1 \right) \]  \hspace{1cm} (2.4)

\[ H_i = \frac{k_b T_p}{m_i g_1} \]

where \( c_i \) is the fractional abundance of the ionic species, \( n \) is electron density, \( R \) is the geocentric distance, \( \Omega \) is the angular speed of rotation of the earth, \( g \) is the acceleration of gravity, \( \theta \) is the dipole latitude, \( k_b \) is the Boltzmann's constant, \( m \) is the mass and \( T_p \) is the plasma temperature. The subscript \( i \) refers to the \( i \)th ionic species and the subscript \( 1 \) refers to the reference level of the diffusive equilibrium model at 1000 km altitude. The coordinates and symbols used are shown in Fig. 2.1. The geopotential height \( z_G \) takes into account the variations of gravitational force with distance and the centrifugal force due to corotation of the plasma with the earth. The diffusive equilibrium model is inaccurate at high latitudes and particularly outside the plasmasphere.

The 'cold' magnetospheric plasma consists of particles with less than a few electron volts energy. The electrons have gyroradii (radius of gyration around the magnetic field) of a few meters. Mixed with the cold plasma are energetic (hot) particles, electrons and positive ions that populate the radiation belts. These particles have energies extending up to hundreds of MeV with electron gyroradii of hundreds of meters. They execute rapid motions along the field lines and stay trapped in the radiation belts due to the inhomogeneity of the earth's magnetic field. The motion of these high energy particles is almost completely
controlled by the magnetic field. The effect of gravitational and centri-
fugal forces can be ignored. In the next section we study the motion of
the energetic particles trapped in the earth's radiation belts.

B. DYNAMICS OF RADIATION BELT PARTICLES

Charge motion in a static magnetic field:

A charged particle moving in the presence of a magnetic field follows
a helical path as a result of its motion parallel to the field superim-
posed on its gyration in the plane perpendicular to the magnetic field,
as shown in Fig. 2.2. The components of the particle velocity parallel
and perpendicular to the field direction are denoted by \( v_\parallel \) and \( v_\perp \)
respectively. The gyrofrequency \( \omega_H \) and the gyroradius \( r_H \) (radius of
gyration around the center line) and the gyroperiod \( T_H \) are given by

\[
\omega_H = \frac{qB_0}{m}; \quad T_H = \frac{2\pi}{\omega_H}
\]

\[
r_H = \frac{m v_\perp}{qB_0} = \frac{m v \sin \alpha}{qB_0}
\]

where \( \alpha = \tan^{-1} \frac{v}{v_\parallel} \) is the particle's pitch angle. Electrons and pro-
tons gyrate in opposite directions as shown in Fig. 2.2.

A static magnetic field cannot do work on the particle since the
force on the particle \((v \times B_0)\) is perpendicular to the direction of motion.
The magnetic field can, however, change the direction of particle velo-
city and therefore the direction of momentum. The total kinetic energy
of the particle, given by \( \frac{1}{2} mv^2 \), is conserved along its trajectory.
Therefore, in a spatially changing magnetic field, such as the earth's
field, the particle orbit (characterized by $\omega_H$ and $r_H$) must change in such a way as to conserve total kinetic energy. The fact that the static magnetic field does no work on the particle means that the magnetic flux linking the orbit of a particle gyrating about a field line (while at the same time moving along the line) is constant. Otherwise in the frame of reference which follows the parallel motion, $\frac{\partial B_0}{\partial t} \neq 0$ and the electric field produced (through $\mathbf{v} \times \mathbf{E} = -\frac{\partial B_0}{\partial t}$) would accelerate the particles, causing a change in energy. Hence,

$$\text{Flux} = B_0 r_H^2 = \text{constant} \quad (2.6)$$
Substituting the expression for $r_H$, we obtain

$$\text{Flux} = B_0 \left( \frac{mv}{qB_0} \right)^2 = \text{constant}$$

since $v = \text{constant}$ (conservation of energy) we have

$$\frac{\sin^2 \alpha}{B_0} = \text{constant} \quad (2.7)$$

Hence as the magnetic field changes along the particle trajectory, the pitch angle must change in accordance with (2.7). Note that we have presented an approximate and heuristic derivation of (2.7). For a precise and rigorous derivation the reader is referred to Northrop [1963] or Buneman [1973]. In order for (2.7) to apply the condition

$$\left( \frac{T_H v}{B_0} \right) \frac{dB_0}{dz} \ll 1 \quad (2.8)$$

where $z$ is the distance along the field line, must hold. In other words the particle must go through many gyrations in a distance over which the magnetic field changes appreciably. This approximation is called the 'adiabatic approximation' and the quantity $\mu = \frac{v^2}{B_0}$ is called the first adiabatic invariant. The condition (2.8) is well satisfied for all inner magnetospheric conditions. According to (2.7), as $B_0$ increases along the trajectory the particle pitch angle also increases. At the point where $\alpha = 90^\circ$, the particle must turn around or 'mirror,' since $v_\parallel$ has been reduced to zero.

In the earth's dipole field as defined by (2.1), it is convenient to relate the mirror point to the equatorial parameters. The value of
the static magnetic field at the mirror point ($\alpha = 90^\circ$) is given by

$$B_m = \frac{B_{eq}}{\sin^2 \alpha_{eq}}$$  \hspace{1cm} (2.9)

where $B_{eq}$, $\alpha_{eq}$ are the magnetic field and the pitch angle at the equator, respectively.

Figure 2.3 shows a portion of the trajectory of a typical particle in a dipole field. Since the dipole field is symmetrical about the magnetic equator the particles mirror at 'conjugate' points as the bounce back and forth between hemispheres.

**Bounce period:**

The time it takes the particle to travel from one mirror point to another is $\tau_B/2$, where $\tau_B$ is called the 'bounce period.' Note that for a given dipole field line (specified by its L value as defined in (2.3)), the location of the mirror point is independent of particle energy and is uniquely determined by $\alpha_{eq}$, through (2.9). The bounce period $\tau_B$ however depends on both $v$ (i.e. energy) and $\alpha_{eq}$, and is given by

$$\tau_B = 2 \int_{z_m}^{z_m'} \frac{dz}{v_{||}(z)}$$  \hspace{1cm} (2.10)

where $z$ is the distance along the field line. We have, at any point,

$$v_{||}(z) = v \cos \alpha(z) = v[1-\sin^2 \alpha(z)]^{1/2} = v[1-\frac{B(z)}{B_{eq}} \sin^2 \alpha_{eq}]^{1/2}$$  \hspace{1cm} (2.11)

where $v$ is the total particle velocity. In the last part of the above we have used (2.7). Now using (2.9) we obtain
FIGURE 2.3 THE TRAJECTORY OF AN ENERGETIC PARTICLE TRAPPED IN THE EARTH'S FIELD.

\[ v_i(z) = v \left[ 1 - \frac{B(z)}{B_m} \right]^{1/2} \]  \hspace{1cm} (2.12)

Substituting in (2.10), we obtain

\[ \tau_B = \frac{2}{v} \int_{z_m}^{z'_m} \frac{dz}{\left[ 1 - \frac{B(z)}{B_m} \right]^{1/2}} \]  \hspace{1cm} (2.13)

This integral can be expressed as [Liemohn, 1961; Hess, 1968; Roederer, 1970]:

- 22 -
\[ \tau_B = \frac{4R \cdot L}{v} \int_0^\lambda \frac{\cos \lambda [1 + 3\sin^2 \lambda]^{1/2}}{\sin^2 \alpha_{eq} [1 + 3\sin^2 \lambda]^{1/2}} \frac{d\lambda}{\cos^6 \lambda} = \frac{4R \cdot L}{v} f(\alpha_{eq}) \quad (2.14) \]

where \( \lambda \) is the latitude. Equation (2.14) can be obtained from (2.13) using the dipole geometry and (2.7) and (2.9). The integrand \( f(\alpha_{eq}) \) has an integrable square root singularity at \( \lambda = \lambda_0 \) (\( \alpha_{eq} \)) and has been evaluated numerically [Liemohn, 1961]. Its value varies from 0.75 for \( \alpha_{eq} = 90^\circ \) to 1.4 for \( \alpha_{eq} = 0^\circ \). A good approximation for \( 40^\circ \leq \alpha_{eq} \leq 90^\circ \) is

\[ f(\alpha_{eq}) \approx 1.30 - 0.56 \sin \alpha_{eq} \quad (2.15) \]

The bounce periods for the radiation belt particles range from one tenth of a second to a few seconds. The gyroperiods of these particles are of the order of \( 10^{-5} \) to \( 10^{-3} \) seconds. Hence the cyclotron (gyro) and bounce motions of these particles have widely different periods and are almost completely separable. This is the basis of the 'adiabatic approximation' the result of which is the expression (2.7).

**Particle precipitation and the loss cone:**

As mentioned above, the particle's mirror point is uniquely determined by \( \alpha_{eq} \). If the mirror point is lowered to atmospheric altitudes (below ~200 km) the particle may collide with the atmospheric neutral constituents and be absorbed. In that case the particle does not bounce back along the field line and is described as being 'precipitated' out of the radiation belts.

At the equator, one can define a minimum value \( \alpha_{eq}^{lc} \) such that for \( \alpha_{eq} < \alpha_{eq}^{lc} \) the particle mirror point would be below some height \( h_m \).
where it will very likely be absorbed by the atmosphere. The pitch angle \( \alpha_{eq}^{1c} \) is called the equatorial 'loss cone' pitch angle and \( h_m \) is called the lowest mirror height. Defining a normalized lowest mirror geocentric radius as

\[
\xi_m = \frac{R + h_m}{R_0}
\]

(2.16)

and using the dipole relations (2.1) through (2.3) and (2.9) we obtain

\[
\sin(\alpha_{eq}^{1c}) = \left[ \frac{\xi_m^3}{L^2/4L^2-3 \xi_m} \right]^{1/2}
\]

(2.17)

It is clear that \( \alpha_{eq}^{1c} \) depends strongly on the particular field line of concern as indicated by the L-value dependence in (2.17). The equatorial loss cone angle as a function of L value for \( h_m = 100 \text{ km} \) and \( h_m = 0 \) is given below.

<table>
<thead>
<tr>
<th>L</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_{eq}^{1c} ) (deg)</td>
<td>16.77°</td>
<td>8.62°</td>
<td>5.47°</td>
<td>3.87°</td>
</tr>
<tr>
<td>( \alpha_{eq}^{1c} ) (deg)</td>
<td>16.33°</td>
<td>8.41°</td>
<td>5.34°</td>
<td>3.78°</td>
</tr>
</tbody>
</table>

As seen above the loss cone angle does not strongly depend on \( h_m \).

In the absence of forces which may alter the equatorial pitch angle the particle will remain trapped in the magnetic field until its azimuthal cross field motion causes it to drift across the field lines toward the sunlit side of the magnetosphere where some particles can become detrapped because of the solar wind induced distortion of the field lines. The cross field drift motion is briefly explained below. This motion is much slower than the bounce motion of the particles. Typical
lifetimes for trapped particles range from an hour to a few days, while some particles may be stably trapped [Roederer, 1970]. Even without any cross-field drifts, a particle's $\alpha_{eq}$ may be altered by wave-particle interactions,* sometimes causing the particle to precipitate. Precipitation induced by wave particle interactions is one of the major loss processes for the radiation belt particles. This report describes precipitation due to one such wave particle interaction, namely the cyclotron resonant interaction involving coherent whistler mode waves.

**Drift across field lines.**

In addition to the gyro and bounce motions the energetic particles execute azimuthal and radial drift motions across the field lines. This is depicted in Fig. 2.4 which shows a geomagnetic equatorial cross section of the earth and the inner magnetosphere.

The important kinds of azimuthal drifts are the gradient-curvature and the $E \times B$ drifts. The gradient-curvature drift is caused by the inhomogeneity of the magnetic field and the curvature of the field lines. In the earth's dipole field this drift is eastward for electrons and westward for protons. The drift periods at $L = 4$ range from -5 days for 10 keV electrons to -15 minutes for 1 MeV electrons. The proton drift periods are smaller than those for the electron but not by more than a factor of 2. The $E \times B$ drift is caused by radial electric fields. This drift is in the same direction for both electrons and protons.

*In a wave particle interaction $\alpha_{eq}$ is modified either because of transfer of energy to (or from) the wave through the wave's electric field, which may change $v_{||}$ or $v_{\perp}$ or because the wave's magnetic field changes the direction of momentum of the particle without much energy transfer.
Radial drifts are mainly $E \times B$ drifts caused by azimuthal electric fields [Carpenter and Seely, 1976]. The drift could be inward or outward depending on the direction of the electric field. Usually radial drifts are slower than azimuthal drifts.

For a more detailed discussion of cross field drifts, the reader is referred to Hess [1968] and Roederer [1970].
C. COHERENT VLF WHISTLER MODE WAVES IN THE MAGNETOSPHERE

The Whistler Mode of Propagation:

This report deals with the cyclotron resonance interaction between energetic radiation belt particles and very low frequency waves which propagate through the magnetospheric plasma in what is commonly known as the "whistler mode." This mode has been discussed in detail by a number of workers [Storey, 1953; Helliwell, 1965]. The plane-wave dispersion relation which describes the mode has been known for many years, having originally been published by Appleton [1927], who in his paper generalized some earlier work on the same topic by Lorentz [1909].

This mode can be identified in the following way. In a cold, collisionless, homogeneous, infinite plasma immersed in a homogeneous static magnetic field, \( \mathbf{B}_0 \), two characteristic electromagnetic waves exist at frequencies below the electron gyrofrequency. In general these characteristic waves are elliptically polarized, the sense of rotation of the wave magnetic vector of one wave being the same as, and the other wave being opposite to, the sense of rotation of the electrons about the static magnetic field lines. The dispersion relation and the polarization for these waves are given by [Ratcliffe, 1959; Budden, 1961; Stix, 1962].

\[
n^2 = 1 - \frac{(f_p^2/f)^2}{1 + \frac{f_p^2 \sin^2 \theta}{2(f_p^2 - f^2)} \left[ \frac{(f_p^2 \sin^2 \theta)}{2(f_p^2 - f^2)} + \frac{f_p^2 H \cos^2 \theta}{f^2} \right]^{1/2}}
\]  

(2.18)
\[ \delta = i \left( \frac{f_H}{f} \cos \theta \right)^{-1} \left\{ 1 + \left[ \frac{f_H^2 \cos^2 \theta}{f^2} + \left( \frac{f_H^2 \sin^2 \theta}{2(f_p^2 - f_H^2)} \right)^2 \right]^{1/2} - \frac{f_H^2 \sin^2 \theta}{2(f_p^2 - f_H^2)} \right\} \]  

(2.19)

where \( n = \frac{k \omega}{c} \) is the refractive index, \( k \) being the wave number, \( \delta \) is the wave polarization, \( \theta \) is the angle between the wave normal and the magnetic field \( B_0 \), \( f \) is the wave frequency, \( f_H \) is the electron gyrofrequency, \( f_p = \frac{1}{2\pi} \frac{n_e e^2}{m_e c} \) is the electron plasma frequency with \( n_e \) being the cold plasma density and \( \varepsilon_0 \) being the permittivity of free space and \( i = \sqrt{-1} \).

In the above equations the choice of the upper sign preceding the radical describes the dispersion of the characteristic wave whose rotational sense is the same as that of the electrons about the field lines, while the choice of the lower sign describes the dispersion of the characteristic wave whose rotational sense is the opposite to that of the electrons.

It is possible to demonstrate that when the square of the plasma frequency is larger than the product of the wave frequency and the sum of the wave and gyrofrequencies, i.e.

\[ \frac{f_p^2}{f} > f(f_p + f) \]  

(2.20)

then the characteristic wave whose rotation is in the opposite sense to that of the gyrating electrons is nonpropagating. In the earth's magnetosphere for \( f < f_H \) the condition (2.20) holds in general and the only propagating wave is one whose sense of rotation is the same as that of the electrons. Under these conditions that characteristic wave is
called the 'whistler mode', the dispersive aspects of which are described by (2.18) with the choice of the upper sign before the radical.

It can be shown that due to condition (2.20), the whistler-mode waves have phase and group velocities that are always less than the velocity of light in vacuum. This is a very important property since it means that such waves can achieve cyclotron or longitudinal resonances with the energetic particles. From (2.18) it can be seen that the phase velocity \( v_p = \frac{\omega}{k} \), where \( k = \frac{n\omega}{c} \) varies with the direction in the plasma indicated by \( \theta \); thus the presence of a static magnetic field causes the medium to be anisotropic. Also the phase velocity (and the group velocity, \( v_g = \frac{d\omega}{dk} \)) depends on the wave frequency \( \omega \) as well as the wave normal \( k \). This means that waves of different frequencies travel with different velocities, i.e. the medium is dispersive.

Equation (2.18) is derived for a homogeneous medium. The magnetosphere is an inhomogeneous medium since both \( f_p \) and \( f_H \) vary in space. However the refractive index generally changes only a small amount in the space of one wavelength. A characteristic wave launched into such a medium exhibits characteristics at a given point that are approximately the same as those of a wave traveling in a homogeneous medium having the same refractive index. A medium having this property is called 'slowly varying'. The corresponding propagation analysis is called 'ray' theory, and the wave functions are called the W.K.B. solutions [Budden, 1961]. Such a slowly varying medium is analogous to a tapered transmission line or waveguide. In order to follow the 'phase path' and 'group path' of a signal propagating in the magnetosphere one needs to use ray tracing.
Equation (2.18) was originally derived for the extremely idealized case of a cold, infinite, collisionless, homogeneous plasma. It is a tribute to the magneto-ionic theory that this equation, together with the W.K.B. solution, has been found to describe quite well the propagation and dispersion of whistlers in the magnetosphere.

Whistlers and Other Coherent VLF Waves

Whistlers are a class of naturally occurring narrow band, audio-frequency electromagnetic radiations. They occur in the frequency range 300 Hz to 30 kHz and are readily detectable at middle and high latitudes by means of an audio amplifier attached to an antenna. These signals are characterized by distinctive variations of frequency with time which will generate at the amplifier a whistling sound, hence the name 'whistler'.

Whistlers result from the dispersion of impulse radiation from lightning strokes in one hemisphere which penetrate the lower ionosphere and travel through the magnetosphere along the earth's magnetic field lines. For frequencies higher than the proton gyrofrequency the dispersion of the original impulsive signal is governed to first order by (2.18) and the impulse is slowly transformed along the path into a smoothly descending (and/or rising) tone, i.e. a whistler, which is received in the conjugate hemisphere.

A portion of the energy of the lightning stroke which causes the whistler may travel in the earth-ionosphere wave guide to the receiver in the conjugate hemisphere. This energy is registered as an impulsive signal and is called the causative atmospheric or 'spheric'.

- 30 -
The frequency-time spectrum of a typical whistler event is shown in the top panel of Fig. 2.5. The time origin of the causative spheric is also shown. The spherics are seen as vertical lines while the several traces of the whistler are seen as quasi-parabolic curves. All traces originate from the same spheric. Three of the traces are sketched in the middle panel. The different traces have propagated on different paths as shown in the bottom panel. Each trace in itself is a narrow-band signal of smoothly changing frequency. Natural whistlers are widely used as a diagnostic tool of the magnetosphere. The frequency of minimum time delay, $f_n$, gives the $L$ value of the path, while, the minimum time delay itself, $t_n$, is a measure of the equatorial cold plasma density [Helliwell, 1965].

The observed dispersion of natural whistlers was found to agree closely with that predicted by using (2.18) along with the W.K.B. approximation [Helliwell, 1965]. This provides strong evidence that the dispersion characteristics of whistler mode waves in the magnetosphere is largely determined by the cold plasma. The energetic plasma can at times cause the waves to be amplified or damped through various instability mechanisms, but their overall effect on the propagation characteristics is negligible.

Whistlers propagate along field-aligned density enhancements in the magnetosphere. These density enhancements are called 'ducts'. The propagation in the ducts is called the 'ducted' mode. The theory of ducted propagation is well developed [Helliwell, 1965]. The presence of the density enhancement provides a variable refractive index in the transverse direction which confines the wave normal to small angles with re-
FIGURE 2.5  A TYPICAL WHISTLER EVENT. This event was observed at Eights (EI), Antarctica, on 6 June 1963 at 0150 Local Time (LT). This time corresponds to 0650:06 Universal Time (UT). The top panel shows the frequency time spectrogram. The middle panel is a sketch of the 'spheric' and three of the many individual traces of the whistler. The lower panel shows the magnetospheric paths traveled by the three traces [Park, 1972].
spect to the static magnetic field. As a result the wave energy is trapped in the duct much like light waves trapped in an optical fiber. Since their wave normal stays close to the magnetic field, ducted waves can penetrate the lower ionospheric boundary and be observed on the ground. For this reason these waves can be studied using receivers or transmitters on the ground. Many passive and active experiments of this kind have been carried out with successive results in the last two decades [Helliwell, 1965; Helliwell and Katsufrakis, 1974; Park and Carpenter, 1977; Park et al., 1977].

Ducts occupy only a small fraction of the inner magnetosphere. Most magnetospheric signals propagate in the 'non-ducted' mode. Nonducted waves do not generally propagate along the static magnetic field, and are not generally received on the ground. They have been seen on various satellites and their study has helped formulate propagation theories in plasmas. Nonducted waves with frequencies below 3 kHz are greatly affected by the presence of various ions in the ionosphere and the magnetosphere. Their dispersion is not simply described by (2.18). The refractive index diagrams are much more complicated, with two possible modes of propagation when the wave frequency lies below the proton gyro-frequency. The study of nonducted VLF and ELF waves or numerous satellites has verified the many cutoffs, resonances and cross overs predicted by the multiple-ion plasma wave propagation theory [Smith and Brice, 1964].

Both ducted and nonducted propagation have been extensively studied using ray tracing. A VLF ray tracing program incorporating multiple-ion plasma wave propagation theory has been developed over the years at
Stanford University [Walter, 1969; Burtis, 1974] and has been successfully used to explain many observed magnetospheric phenomena [Kimura, 1966; Smith and Angerami, 1968; Scarabucci, 1969; Walter, 1969; Angerami, 1970; Edgar, 1972; Inan et al., 1977b]. One recent application of ray tracing showed that the large density gradients of the plasmapause (see Fig. 1.2) can guide VLF waves in a manner very similar to normal ducting [Inan and Bell, 1977].

The success in the last two decades of the passive studies of the natural whistler events has led to active probing of the magnetosphere involving VLF wave-injection experiments using ground transmitters such as that at Siple Station, Antarctica [Helliwell and Katsufrakis, 1974]. This experiment involves a 100 kW VLF transmitter at Siple and a VLF receiver at the conjugate point in Roberval, Canada. In Figs. 2.6, 2.7, and 2.8 we show coherent VLF waves injected into the magnetosphere by the Siple VLF transmitter. Detailed explanations are given in the figure captions. Note that these signals are all coherent narrow-band signals with bandwidths less than 50 Hz.

Another source of coherent VLF waves in the magnetosphere is the harmonic radiation from large scale power grids [Helliwell et al., 1975; Park, 1976]. These signals propagate in the magnetosphere and are received at the conjugate point. Figure 2.9 shows an example of these signals.

In addition to the examples of coherent VLF waves just discussed, the magnetosphere contains various kinds of 'incoherent' noise, usually called hiss because of the hissing sound they generate when applied to an audio amplifier. Figure 2.10 shows an observation on the OGO-4 satel-
FIGURE 2.6 OBSERVATIONS OF TRANSMITTER SIGNALS ON THE IMP-6 SATELLITE [Inan et al., 1977a]. The bottom panel shows the format of transmissions as they are recorded at the transmitter site at Siple, Antarctica. The format in this case is frequency shift keying (FSK) with one-second long pulses at two frequencies. The top panel shows the reception on the satellite. Note the time delay of -0.3 seconds. Also note the reception of a whistler trace in the middle.
FIGURE 2.7 TRANSMITTER SIGNALS RECEIVED AT THE CONJUGATE POINT AT ROBERVAL, CANADA. The transmitter format is shown below. Note that the first 2-second long pulse has triggered strong emissions. The theory of triggered VLF emissions is not fully understood, although it is thought that the changing frequency is caused by the inhomogeneity of the magnetic field [Helliwell, 1967, 1970]. Note that although the emissions change frequency with time, they are still quite narrowband and coherent signals. The second pulse (4-second long) has not triggered emissions and it does not show any large amplification. For the parameters used in our computations, a 4-second pulse is approximately equivalent to a CW.
FIGURE 2.8 RECEPTION OF TRANSMITTER SIGNALS [Helliwell and Katsurfrakis, 1977]. The bottom panel shows the transmitted format which in this case is a staircase pattern. The receptions at Roberval at two different times are given. Note that the transmitter signals trigger falling and inverted hook emissions [Helliwell, 1965], that are entrained by the next pulse. Also note the amplification of the pulses before triggering. The pulses in the last portion of the middle panel do not show any triggering.
FIGURE 2.9  EXAMPLE OF POWER LINE RADIATION INTO THE MAGNETOSPHERE
[Helliwell et al., 1975]. Simultaneous spectra from the conjugate stations Siple and Roberval are given. The broad horizontal lines on the spectra are spaced roughly 120 Hz apart and are closely associated with the relatively thin induction lines shown on the spectrum from Roberval. Some of these induction lines are identified on the right hand side by their harmonic order. Note that these are due to the Canadian power grid and therefore they are harmonics of 60 Hz. Note that the induction lines are not present at Siple although the magnetospheric lines radiated by the power grid are. The small separation between the magnetospheric lines and the induction lines is thought to result from the triggering of rising emissions by the actual power line harmonic component. These emissions form a somewhat irregular band just above the induction power lines.
FIGURE 2.10  SPECTROGRAM OF A TYPICAL EXAMPLE OF OBSERVATION OF ELF HISS ON THE OGO-4 SATELLITE [Muzzio, 1971]. The reception as the satellite moves from -5° to beyond -30° dipole latitude is shown. Note that the hiss is an incoherent, broadband signal with a well defined lower cutoff frequency. Note from the upper panel that the upper cutoff varies from ~400 Hz to ~800 Hz in one minute corresponding to a latitude range of ~3.5°.
lite of ELF hiss as an example of such a signal. The details are described in the figure caption.

**Longitudinal Whistler Mode Waves**

In our simulation of the cyclotron resonance wave particle interaction we will consider the case of the wave propagation directly along the ambient magnetic field lines, i.e. \( \mathbf{k} \parallel \mathbf{B}_0 \). Exact longitudinal propagation is unlikely to occur, of course, but in light of the duct theory of whistler propagation [Smith, 1961], it is a reasonable approximation. This assumption considerably simplifies the equations of motion.

For \( \mathbf{k} \parallel \mathbf{B}_0 \), i.e. \( \theta = 0 \), Eqs. (2.18) and (2.19) reduce to

\[
n^2 = 1 - \frac{f_p^2}{f(f_H - f)} \quad (2.21)
\]

and

\[
\delta = \pm i \quad (2.22)
\]

In the inner magnetosphere \( f_p^2 \gg f_H^2 \), and for \( f < f_H \) the choice of the lower sign in (2.21) results in \( n^2 < 0 \), i.e. a nonpropagating wave. In light of this and using \( f_p^2 \gg f_H^2 \), we can rewrite the refractive index and wave polarization for longitudinal whistler mode waves in the earth's magnetosphere

\[
n = \frac{k c}{2 \pi f} \approx f_p \left[ \frac{1}{f(f_H - f)} \right]^{1/2} \quad (2.23)
\]

and

\[
\delta = \pm i \quad (2.24)
\]

where \( c \) is the speed of light and \( k \) is the wave number in the medium.

Other useful expressions that can be derived from (2.23) are given below.
\[ k = \frac{2\pi f}{c} \left( \frac{f}{f_H-f} \right)^{1/2} \]  
(2.25)

\[ v_p = \frac{\omega}{k} = c \frac{f^{1/2}(f_H-f)^{1/2}}{p_{f_H}} \]  
(2.26)

\[ v_g = \frac{d\omega}{dk} = 2c \frac{f^{1/2}(f_H-f)^{3/2}}{p_{f_H}} \]  
(2.27)

where \( v_g \) is the group velocity which is equal to signal propagation velocity for longitudinal propagation.

Equation (2.24) indicates that these waves are right hand circularly polarized, i.e. the wave vectors rotate in the same sense as electrons about the field lines. Both the wave electric and magnetic fields are transverse to \( \overrightarrow{B_0} \). The wave vectors traverse a helix in space as shown in Fig. 2.11. Since \( \overrightarrow{E_w} \parallel \overrightarrow{k} \), \( \overrightarrow{k} \cdot \overrightarrow{E_w} = 0 \) and we have \( \rho_s = 0 \), which means that the wave will propagate without space-charge. The plasma manifests itself via currents which flow in planes normal to \( \overrightarrow{k} \) and \( \overrightarrow{B_0} \).

The expressions for \( \overrightarrow{E_w} \) and \( \overrightarrow{B_w} \) given in Fig. 2.7 are for propagation in a homogeneous medium. Although the magnetospheric medium is inhomogeneous the variations in \( \omega_H \) and \( \omega_p \) within the space of one wavelength are negligible. Therefore, the W.K.B. approximation [Budden, 1961] can be used and the wave fields can still be expressed as progressive waves as follows:

\[ \overrightarrow{B_w} = B_w \left[ a_x \cos(\omega t - \int_0^z k\,dz) + a_y \sin(\omega t - \int_0^z k\,dz) \right] \]  
(2.28)

\[ \overrightarrow{E_w} = \frac{\omega}{k} B_w \left[ a_x \sin(\omega t - \int_0^z k\,dz) - a_y \cos(\omega t - \int_0^z k\,dz) \right] \]
FIGURE 2.11 LONGITUDINAL WHISTLER-MODE WAVE FIELDS. Helical locus of the wave magnetic field is shown. The corresponding wave electric field locus is a similar helix but 90° out of phase such that at each point $\vec{E}_w \times \vec{B}_w$ is in the direction of $\vec{k}$. The wave fields can be expressed as:

$$\vec{B}_w = B_w \left[ \vec{a}_x \cos(\omega t - kz) + \vec{a}_y \sin(\omega t - kz) \right]$$

and

$$\vec{E}_w = E_w \left[ \vec{a}_x \sin(\omega t - kz) - \vec{a}_y \cos(\omega t - kz) \right]$$
In the next section we develop the equations of motion for energetic particles in a longitudinal whistler mode wave as described above.

D. EQUATIONS OF MOTION

In this section we shall derive and discuss the equations that govern the motion of an energetic particle in a longitudinal whistler mode wave.

The basic assumption that has been experimentally verified as mentioned in previous sections is that the energetic particle motions often do not significantly affect the propagation of the wave or each other. This assumption is justified by the fact that the total density of the energetic particles is much less than the cold plasma density. The wave's dispersion and propagation characteristics are governed entirely by the 'cold' plasma which can be safely assumed to be collisionless. The refractive index, $n$, is then given by (2.23) and the cold plasma acts only as a slow wave structure for the whistler mode, similar to slow wave structures in microwave tubes. The energetic particles can then be considered as a separate 'beam' of electrons, again similar to electron beams in microwave tubes. The problem is more complex because of the transverse gyration as well as longitudinal motions of both the wave vectors and the electron beam. On the other hand there is no significant space-charge bunching involved in the longitudinal cyclotron resonance mechanism.

In order for the wave to induce cumulative energy and/or momentum changes in the particle the wave vectors as 'seen' by the particle must be approximately stationary for a significant length of time. This means that the doppler shifted frequency as 'seen' by the particle must be
approximately equal to its gyrofrequency, i.e.

\[ \omega - \vec{k} \cdot \vec{v} = \omega_H \]  

(2.29)

This is the cyclotron (or gyro) resonance condition. For the whistler mode, where \( \omega < \omega_H \), (2.29) can be satisfied only if \( \vec{k} \cdot \vec{v} < 0 \) or when the resonant electrons and the wave travel in opposite directions. We can then rewrite (2.29) as

\[ \omega + kv_H = \omega_H \]  

(2.30)

where \( v_H \) is the particle velocity along \( -\vec{B}_0 \) as shown in Fig. 2.12.

In the absence of the wave the energetic particles are trapped along a given field line and their unperturbed motion, neglecting the small azimuthal drift, can be described by the following relations which can be derived from the adiabatic invariant relation (2.7):

\[
\begin{align*}
\left( \frac{dv_H}{dt} \right)_u &= - \frac{v_H^2}{2B_0} \frac{dB_0}{dz} \\
\left( \frac{dv_\perp}{dt} \right)_u &= + \frac{v_H v_\perp}{2B_0} \frac{dB_0}{dz}
\end{align*}
\]  

(2.31)

The coordinate system and the variables are defined in Fig. 2.12.

The force \( \vec{F} \) applied to a particle by the wave is given by

\[ \vec{F} = q(\vec{E}_w + \vec{v} \times \vec{B}_w) \]  

(2.32)

where \( \vec{v} = v_H + \vec{v}_\perp \) is the total particle velocity. Separating \( \vec{F} \) into its components parallel and perpendicular to \( \vec{B}_0 \) we have
FIGURE 2.12  COORDINATE SYSTEM FOR THE EQUATIONS OF MOTION. Note that the z-axis is everywhere aligned with the magnetic field line. Shown in dashed lines is the orbit of the electron in the x-y plane.
\[
\vec{F} = \vec{F}_\parallel + \vec{F}_\perp = -e[\vec{E}_w \times \vec{B}_w + \vec{V}_w \times \vec{B}_w]
\]  
(2.33)

Since both \(\vec{E}_w\) and \(\vec{B}_w\) are perpendicular to \(\vec{k}\) and \(\vec{B}_0\) (2.33) can be expressed as

\[
\vec{F}_\parallel = -e \vec{V}_w \times \vec{B}_w
\]  
(2.34a)

\[
\vec{F}_\perp = -e \vec{E}_w - e \vec{V}_w \times \vec{B}_w
\]  
(2.34b)

The force \(\vec{F}_\parallel\) provides an acceleration along \(\vec{V}_\parallel\), whereas \(\vec{F}_\perp\) is in general in the same plane but at some angle with \(\vec{V}_\perp\). With the coordinate system as given in Fig. 2.12 and using (2.34a,b) we can write the wave induced accelerations along \(\vec{V}_\parallel\) and \(\vec{V}_\perp\) as:

\[
\left(\frac{dv_\parallel}{dt}\right)_{\text{wave}} = -\frac{e}{m} |\vec{V}_w| \times |\vec{B}_w| = \left(\frac{e}{m} \omega\right) v_\parallel \sin \phi
\]  
(2.35a)

\[
\left(\frac{dv_\perp}{dt}\right)_{\text{wave}} = -\frac{e}{m} |\vec{E}_w| \sin \phi - \frac{e}{m} |\vec{V}_w| |\vec{B}_w| \sin \phi = \left(-\frac{e}{m} \omega\right) \sin \phi - \left(\frac{e}{m} \omega\right) v_\parallel \sin \phi
\]  
(2.35b)

where \(\phi\) is the angle between \(\vec{V}_\perp\) and \(-\vec{B}_w\) as shown in Fig. 2.12.

Note that the differential equations (2.35), when integrated properly completely describe the wave's effect on the particle velocity components \(v_\parallel\) and \(v_\perp\). Note also that both \(\frac{dv_\parallel}{dt}\) and \(\frac{dv_\perp}{dt}\) are proportional to \(\sin \phi\), meaning that whether or not there is any integrated cumulative change in \(v_\parallel\) and \(v_\perp\) depends on the variation of \(\phi\). The rate of change of \(\phi\) is given by
\[
\frac{d\phi}{dt} = (\omega_H - \omega - k \nu) - \left( \frac{eB_w}{m} \right) \nu \cos \phi - \left( \frac{eE_w}{m} \right) \cos \phi \frac{\nu}{\nu}.
\] (2.35c)

The first term on the right hand side of the above represents the difference between the doppler shifted wave frequency seen by the particle and its own gyrofrequency, which by definition is the rate of change of \( \phi \) to first order. The last two terms give the phase change due to centripetal acceleration of the particle resulting from \( \nu \times B_w \) and \( E_w \) forces. This centripetal acceleration modifies the particle motion about its guiding center so that the angular frequency of oscillation about the static magnetic field deviates slightly from the gyrofrequency, \( \omega_H \).

Equations (2.35a,b) give only the wave induced variations in \( \nu \) and \( \nu \), However these quantities also vary due to the inhomogeneity of the ambient field as given by (2.7) and((2.31). Superimposing these on (2.35a,b), using \( |E_w| = j \omega K_w \) and combining terms we obtain the equations of motion,

\[
\dot{\nu} = \left( \frac{eB_w}{m} \right) \nu \sin \phi - \left( \frac{\nu}{2\omega_H} \right) \frac{d\nu}{dz} (2.36a)
\]

\[
\dot{\nu} = - \left( \frac{eB_w}{m} \right) (\nu + \frac{\omega}{k}) \sin \phi + \left( \frac{\nu}{2\omega_H} \right) \frac{d\omega}{dz} (2.36b)
\]

\[
\dot{\phi} = \omega - \omega - k \nu = \left( \frac{eB_w}{m} \right) (\nu + \frac{\omega}{k}) \cos \phi \frac{\nu}{\nu} (2.36c)
\]

where the "\( \dot{\cdot} \)" represents the derivative with respect to time. Since the medium is inhomogeneous the quantities \( \omega \) and \( k \) are functions of \( z \), the distance along the field line. Equations (2.36) are written in the laboratory frame, with the coordinate system as described in Fig. 2.12. For our purposes no important simplification results by trans-
forming to either the wave or the particle frames.

Equations (2.36) in whole or in part have been used to study VLF wave-particle interactions by many authors [Bell, 1964; Brice, 1964; Bell, 1965; Helliwell, 1967, 1970; Dysthe, 1971; Nunn, 1971, 1974; Palmadesso and Schmidt, 1971, 1972; Matsumoto, 1972; Ashour-Abdalla, 1972; Bud'ko et al., 1972; Crystal, 1973; Helliwell and Crystal, 1973; Roux and Pellat, 1976; Karpman et al., 1974a,b]. Dysthe [1971] was the first to include the adiabatic terms in Eqs. (2.36a) and (2.36b).

The entire physics of the interaction is embodied in Eqs. (2.36a,b and c). For this reason a brief discussion of the importance of different terms in (2.36) is in order. The first terms in Eqs. (2.36a) and (2.36b) are due to the wave induced longitudinal and transverse forces \( \mathbf{v}_\perp \times \mathbf{B}_w \) and \( \mathbf{w} \times \mathbf{B}_w + \mathbf{E}_w \) respectively. Also both of these equations have an additional term which gives the adiabatic variations of \( \mathbf{v}_\parallel \) and \( \mathbf{v}_\perp \) that are superimposed on the wave perturbations. It should be noted that although the presence of field aligned wave forces violates the assumptions underlying the first adiabatic invariant [Roederer, 1970], the changes in \( \mathbf{v}_\parallel \) and \( \mathbf{v}_\perp \) due to changing static magnetic field intensity can still be described by the differential adiabatic theory at each point during the interaction. In other words, Eqs. (2.36) are differential equations valid at each point along the field line.

Since the wave terms in Eqs. (2.36a) and (2.36b) are proportional to \( \sin \phi \) it is apparent that the interaction is strongly controlled by the third equation (2.36c) which gives the variation of \( \phi \). Cumulative changes in \( \mathbf{v}_\parallel \) and \( \mathbf{v}_\perp \) will only result when \( \dot{\phi} \) (and \( \ddot{\phi} \)) is small. The wave term in Eq. (2.36c) (last term on the r.h.s.) gives the phase
change due to the centripetal acceleration of the particle resulting from the \((\mathbf{v}_\perp \times \mathbf{B}_\perp + \mathbf{E}_\perp)\) force. We have found that for most magnetospheric parameters the effect of this term is negligible, especially for large pitch angles \((\text{large } v_\perp)\) and/or small wave amplitudes. However, even for small wave intensities this wave term becomes dominant as soon as the pitch angle falls below 1 or 2 degrees. Note that individual particles that have somewhat larger initial pitch angles could still be scattered down to these low pitch angles during the interaction. At that time this term must be present in the equations in order to correctly describe the physics.

The Eqs. (2.36) define the motion of each individual electron. For a particle with initial velocities of \(v_\parallel_0\) and \(v_\perp_0\) and an initial phase of \(\phi_0\), these equations, when properly integrated over time, give the resulting \(\Delta v_\parallel\) and \(\Delta v_\perp\) for that particle. In other words

\[
\Delta v_\parallel = \int_0^{T_I} \mathbf{v}_\parallel \, dt
\]

\[
\Delta v_\perp = \int_0^{T_I} \mathbf{v}_\perp \, dt
\]

where \(T_I\) is the interaction time. In principle, the integration must be carried out for \(T_I \to \infty\). However because of the changing \(\omega_H\) due to the inhomogeneity of the field the resonance condition (2.30) can be satisfied only over a limited region along the field line for any given particle. When (2.30) is not satisfied, it is apparent from Eq. (2.36c) that \(|\phi|\) would be large and the wave contributions to \(\mathbf{v}_\parallel\) and \(\mathbf{v}_\perp\) will be noncumulative. Therefore a time \(T_I\) can be defined over which
the wave induced perturbations are significant.

As a result of its encounter with the wave, the particles $v_\parallel$ and $v_\perp$ changes. These changes $\Delta v_\parallel$ and $\Delta v_\perp$ can be viewed as a scattering in velocity space and are hereafter referred to as "scattering."

As is apparent from the differential equations (2.36) the point-to-point scatterings (and therefore the integrated total scatterings) depend on $v_{0\parallel}$ and $v_{0\perp}$, as well as on the initial phase $\phi_0$ between $v_\perp$ and $-B_w$ at the time of encounter between the wave and the particle. The scattering also depends on wave parameters $B_w$, $\omega$ and $k$ and medium parameters $\omega_H$ and $\frac{\partial \omega}{\partial z}$. In general when the resonance occurs away from the equator, for constant frequency, the cumulative interaction time is shortened (since $\frac{\partial \omega}{\partial z}$ is larger) and the total scattering is smaller. The quantity $\frac{\partial \omega}{\partial z}$ is the principal factor which determines the interaction time $T_z$.

In order to clearly see the effect of $\frac{\partial \omega}{\partial z}$ we follow a procedure similar to that used by Dysthe [1971]. Neglecting the wave terms on the r.h.s. of Eq. (2.36c) we can rewrite it as:

$$\dot{v} = \omega - k v_\parallel = k(v_R - v_\parallel)$$

(2.38)

where $v_R = \frac{\omega - \omega}{k}$ is the resonance velocity obtained from (2.30). Differentiating the above we obtain

$$\ddot{v} = k(\dot{v}_R - \dot{v}_\parallel) + \dot{k}(v_R - v_\parallel)$$

(2.39)

for monochromatic waves ($\dot{\omega} = 0$). Since we are concerned with resonant or closely resonant particles, $v_\parallel \approx v_R$ and to a good approximation (2.39) reduces to

$$- 50 -$$
\[
\ddot{\phi} = k(\dot{v}_R - \dot{v}_\parallel)
\]  
(2.40)

Substituting \(\dot{v}_\parallel\) from Eq. (2.36a) we have

\[
\ddot{\phi} = kv_R - k \left( \frac{eB_w}{m} \right) v_\perp \sin \phi + \frac{k v^2}{2 \omega_H} \frac{\partial \omega_H}{\partial z}
\]  
(2.41)

We have

\[
\dot{v}_R = \frac{d}{dt} \left[ \frac{\omega_H - \omega}{k} \right] = \frac{\omega_H - (\omega_H - \omega)k}{k^2}
\]  
(2.42)

Rewriting Eq. (2.25) for \(k\),

\[
k = \frac{\omega_H^{1/2}}{c} \left( \frac{\omega_H - \omega}{\omega_H - \omega} \right)^{-1/2}
\]  
(2.43)

In a diffusive equilibrium model of the cold plasma, such as the one used in this report, the plasma frequency \(\omega_p\) is approximately constant along the field line. Hence \(\dot{\omega}_p = 0\), and we obtain

\[
\dot{k} = \left( -\frac{1}{2} \right) \frac{\omega_p}{c} \left( \frac{\omega_H - \omega}{\omega_H - \omega} \right)^{-3/2} \omega_H = -\frac{k}{2(\omega_H - \omega)} \omega_H = -\frac{k v_H}{2(\omega_H - \omega)} \frac{\partial \omega_H}{\partial z}
\]  
(2.44)

Substituting in (2.42) and then (2.41) and rearranging terms we obtain

\[
\ddot{\phi} + k \left( \frac{eB_w}{m} v_\perp \right) \sin \phi = \left[ \frac{3}{2} \frac{k v_H}{2 \omega_H} \right] + \frac{\partial \omega_H}{\partial z}
\]  
(2.45)

Equation (2.45) is a type of 'pendulum' equation which gives the variation of the phase \(\phi\). The forcing function of this equation is proportional to \(\frac{\partial \omega_H}{\partial z}\), hence demonstrating the influence of the term \(\frac{\partial \omega_H}{\partial z}\), i.e. the inhomogeneity force, in controlling the interaction. The total interaction time \(T_I\) is determined by the relative magnitudes of this forcing function and the restoring 'force' (the wave force) of the pendulum which is proportional to \(k \left( \frac{eB_w}{m} v_\perp \right)\).
E. ENERGY AND MOMENTUM TRANSFER

Equations (2.36) are in terms of the velocity components $v_\parallel$ and $v_\perp$. This formulation is useful, since the resonance condition (2.30) is in terms of $v_\parallel$. However, in order to see the energy and pitch angle changes of the particle more clearly it is convenient to express the equations of motion in terms of $v$ and $\alpha$.

Using $v_\parallel = v \cos \alpha$ and $v_\perp = v \sin \alpha$ and Eqs. (2.7), (2.31) and (2.36) we obtain

$$\frac{dv}{dt} = -(E_x/m) \sin \alpha \sin \phi$$  

(2.46a)

$$\frac{d\alpha}{dt} = -(E_y/m) \sin \phi - \frac{E_x}{m} \frac{\sin \alpha}{v} \sin \phi + \frac{v \sin \alpha}{\omega_H} \frac{d\omega_x}{dz}$$

(2.46b)

$$\frac{d\phi}{dt} = \omega_H - k v \cos \alpha - \left( \frac{E_x}{m} \frac{\cos \alpha}{\sin \alpha} \cos \phi - \left( \frac{E_y}{m} \frac{\cos \alpha}{v \sin \alpha} \cos \phi \right)$$

(2.46c)

In the above, the last term in (2.46b) is the equivalent of the last terms in (2.36a,b). Note that Eq. (2.46a) directly gives the energy change of the particle, since the total particle kinetic energy is $K_E = 1/2 mv^2$. The energy exchange can only occur through the electric field of the wave since magnetic forces are always perpendicular to the direction of motion. Since the $E_x$ for our case is purely transverse, energy exchange can only occur through $v_\perp$. For this reason any energy exchange will also result in pitch angle change as shown by the $E_x$ term in Eq. (2.46b). Although $E_x$ cannot induce energy change, it is apparent from (2.46b) that it causes a pitch angle change. The pitch angle change is in essence a change in the direction of momentum (or velocity) of the particle.
Considering only the wave perturbations, i.e. neglecting the adiabatic term in Eq. (2.46b) and using \( E_w = \frac{\omega}{k} B_w \) we find from (2.46a,b) that

\[
\frac{dx}{dv} = \frac{\left[ k + \cos \alpha \right]}{\sin \alpha} \frac{\omega}{v}
\]

(2.47)

In order to compare normalized pitch angle changes to normalized changes in energy we can form the following ratio using (2.47):

\[
\eta = \frac{\frac{dx}{dv}}{\frac{dK_E}{K_E}} = \frac{\left[ \frac{v}{v_p} + \cos^2 \alpha \right]}{\alpha \sin 2\alpha}
\]

(2.48)

where \( v_p \) is the wave phase velocity.

Since only the particles at or close to resonance with the wave will suffer significant scattering we use (2.30) in conjunction with (2.25) and (2.26) to obtain

\[
\frac{v}{v_p} = \frac{f_H - f}{f}
\]

(2.49)

Substituting this in (2.48) we have

\[
\eta = \frac{\frac{dx}{dv}}{\frac{dK_E}{K_E}} = \frac{\left[ \frac{f_H - f}{f} + \cos^2 \alpha \right]}{\alpha \sin 2\alpha}
\]

(2.50)

In (2.50) \( \eta \) is independent of the plasma frequency \( f_p \). We have plotted (2.50) against pitch angle for typical values of \( f_H = 13.65 \text{ kHz} \) at the equator on the \( L = 4 \) field line) and a few different typical values of wave frequency \( f \) in Fig. 2.13. Since we have used the value of \( f_H \) at the equator the results shown are only for resonances at the
Figure 2.13 The quantity, $\eta$, defined as the ratio of percentage pitch angle change to percentage energy change as a function of pitch angle. The results plotted are for $f_H = 13.65$ kHz corresponding to equator on the $L = 4$ field line. A few different typical values for the wave frequency is used.
equator. For off-equatorial resonances $f_H$ is larger and therefore $\eta$ would be even higher. Figure 2.13 shows that $\eta >> 1$ for most cases and $\eta > 1$ for all cases considered. This means that large percentage pitch angle changes can be induced without much energy transfer. Hence even a weak wave could produce significant pitch angle perturbations. Note for example that the loss cone at $L = 4$ is about 5.5° wide. Therefore to precipitate a particle with $\alpha = 10°$ into the loss cone requires ~50% pitch angle change. Since $\eta = 40$ from Fig. 2.13, this requires only a 1.2% change in energy.

It will be shown in the last sections of this report that, consistent with the arguments above, the total precipitated energy flux is 50 dB higher than the wave input energy. The wave induced precipitation occurs through change of direction of momentum of the particles rather than through energy exchanges. In this particular gyroresonant wave-particle interaction where the whistler mode wave involved is a 'slow' wave with $v_p = 0.01c-0.1c$, wave magnetic field effects upon particle motion are generally more important than the wave electric field effects. The wave magnetic field cannot induce energy exchange, but it can change the particle pitch angle, i.e. direction of momentum.

F. LINEAR THEORY

Later in this report, we will be comparing our results with linear theory which has been used by previous workers in this field. In Chapter 4 we derive a quantitative criterion for determining the applicability of linear theory under given conditions. For these reasons we devote this section to a brief discussion of linear theory.
The basic problem is the integration of Eqs. (2.36a,b) in order to obtain the perturbed values of $\psi_\parallel$ and $\psi_\perp$. One common approach is to use linear theory [Das, 1971; Ashour-Abdalla, 1972]. According to this theory the wave induced scatterings, $\Delta\psi_\parallel$ and $\Delta\psi_\perp$, are computed by using wave field components at the position of the particle as given by the unperturbed ($B_w = 0$) motion. In other words it is assumed that the wave effects are so small that the variation of $\phi$ (as given by (2.36c)) is very close to what it would have been for $B_w = 0$. In that case, the starting point is

$$\dot{\phi} = \omega_H - \omega - kv_\parallel = f(t) \quad (2.51)$$

where $f(t)$ is some function of $t$ which is defined by the variation of $\omega_H$ and $v_\parallel$ along the particle trajectory. Then

$$\phi = \phi_u = F(t) + \phi_0 \quad (2.52)$$

where $F(t) = \int_0^t f(t')dt'$. The function $F(t)$ does not depend on wave intensity and is independent only on $\omega_H$, $\frac{\partial \omega_H}{\partial z}$, $k$ and pitch angle. Equation (2.52) thus gives the unperturbed ($\phi_u$) variation of the phase. Using analytical models for $\omega_H(z)$ and $k(z)$ it is possible to find an expression for $\phi_u$ [Helliwell, 1970].

When this unperturbed variation for $\phi$ is used the integration of (2.36a,b) becomes relatively easy. The linear theory calculation of $\Delta\psi_\parallel$ and $\Delta\psi_\perp$ then proceeds by substituting $\phi_u$ into Eq. (2.36a,b) and integrating.
\[
\Delta \mathbf{v} = \int_{0}^{\infty} \left( \frac{eB}{m} \right) \mathbf{v}_\perp \sin[F(t) + \phi_0] dt + \left( \frac{eB}{m} \right) \mathbf{v}_\parallel \int_{0}^{\infty} \sin[F(t) + \phi_0] dt
\]
\[
\Delta \mathbf{v}_\perp = -\int_{0}^{\infty} \left( \frac{eB}{m} \right) (\mathbf{v}_\parallel + \frac{\omega}{k}) \sin[F(t) + \phi_0] dt - \left( \frac{eB}{m} \right) (\mathbf{v}_\parallel + \frac{\omega}{k}) \int_{0}^{\infty} \sin[F(t) + \phi_0] dt
\]

(2.53a)

(2.53b)

where the adiabatic terms in (2.36a,b) have been dropped. In the above equations \( \Delta \mathbf{v} \) and \( \Delta \mathbf{v}_\perp \) are the total changes in \( \mathbf{v} \) and \( \mathbf{v}_\perp \). In moving \( \mathbf{v}_\perp \) and \( (\mathbf{v}_\parallel + \frac{\omega}{k}) \) out of the integral in (2.53a) and (2.53b) respectively, we have used the linear theory assumption that the adiabatic variations in \( \mathbf{v}_\perp \) and \( \mathbf{v}_\parallel \) are small during the time the particle is close to resonance, and that \( \Delta \mathbf{v}_\perp \ll \mathbf{v}_\perp \) and \( \Delta \mathbf{v}_\parallel \ll \mathbf{v}_\parallel \). The former assumption is valid unless the initial pitch angle is very large or the resonance point is at high latitudes. The latter is a requirement of linear theory and is consistent with the small wave intensities that justify the application of this theory.

With certain simplifying assumptions the integrations in (2.53a) and (2.53b) can be carried out and analytical expressions for \( \Delta \mathbf{v}_\parallel \) and \( \Delta \mathbf{v}_\perp \) can be obtained [Ashour-Abdalla, 1972].

Although it is generally agreed that linear theory is applicable for sufficiently low wave intensities, none of the authors who have employed this theory in their analyses has given quantitative criteria to justify their assumptions. In Chapter 4 we develop a simple quantitative criteria that can be used to determine the applicability of linear theory under given conditions. Qualitatively, when the wave intensity is large, the wave induced variations in \( \mathbf{v}_\parallel \) are significant. This couples to \( \phi \).
through $v_\parallel$ in Eq. (2.36c) and causes $\phi(t)$ to deviate from $\phi_\parallel$.

Therefore linear theory cannot be used and a complete integration of the coupled equations (2.36) must be carried out.

Integration of these highly nonlinear equations is best done on a digital computer. In the next chapter we describe our computer simulation of the equations of motion.
III. DESCRIPTION OF THE SIMULATION

A. INTRODUCTION

In this chapter we describe the method of solution employed in this report to compute the wave induced perturbations of the energetic particles. The approach used is a test particle simulation of the wave particle interaction. The perturbation of a full particle distribution is calculated by considering the effect of the wave on a sufficiently large number of test particles that are appropriately distributed in the phase space. With this point of view the problem becomes one of classical Newtonian mechanics. Namely, given a wave structure, the task is to simulate the equations of motion for the individual test particles. With the computer simulation approach it is possible to use the full equations of motion and to test quantitatively the relative importance of different terms. The need for restrictive simplifying assumptions is also greatly reduced.

The modeling of a full particle distribution is described in Chapter 5. In this chapter we describe in detail the computation of the trajectory of a given particle in the distribution.

A simplified version of the computer program used in our calculation are given in Appendix D. In the following we describe the general formulation of the problem.

B. COMPUTATION OF THE MEDIUM PARAMETERS

The computer simulation employs a centered dipole model for the static magnetic field described by Eq. (2.1) and a diffusive equilibrium model described by Eq. (2.4) for the cold plasma density. The computa-
tion of the medium parameters is separated from the simulation of the interaction. For this reason any other cold plasma model can be used without difficulty.

The necessary input to this portion of the program consists of wave frequency $\omega$, specification of the field line by its $L$ value and the equatorial cold plasma density $n_{eq}$, on that line. Another input is a latitude range, $\text{YLAMAX}$, which defines the portion of the field line within which the calculation is to be limited. For any given parameters a value of $\text{YLAMAX}$ can be found such that for gyroresonant interactions at latitudes greater than $\text{YLAMAX}$ the particle perturbations are negligible or below the resolution of the computations. The simulation is then limited to within $\pm\text{YLAMAX}$ of the equator. Another input to this portion of the program is an integer value $M$. This value specifies the number of grid points that is to be taken along one half of the field line. With these inputs specified the program then computes $\omega_H$, $\omega_p$ and $k$ and stores the values $\omega_H(z)$ and $k(z)$ at $M$ points equally spaced in latitude between $-\text{YLAMAX}$ and zero degrees (equator) latitude. This is illustrated in Fig. 3.1. Hence, the output of this portion of the program is $M$ triplets $z$, $\omega_H(z)$ and $k(z)$, where $z$ is the distance along the field line as measured from the equator. Typical values used in our calculations are $\text{YLAMAX} = 20^\circ$ and $M = 10,000$.

These stored values of $\omega_H(z)$ and $k(z)$ are used later during the integration of the motion equations.

C. MAIN SIMULATION

**Test particle identification**

In a test particle simulation each particle must be identified with
Figure 3.1  COMPUTATION OF THE MEDIUM. The portion of the field line to which the computation is limited is divided into mesh points equally spaced in latitude. For every mesh point the triplet $z, \omega_H(z)$ and $k(z)$ is stored for use in the integration of the motion equations.

A set of unique parameters. For adiabatically trapped particles the equatorial pitch angle, $\alpha_{\text{eq}}$, and equatorial parallel velocity $v_{\parallel \text{eq}}$ are one such set of quantities. Using the first adiabatic invariant (Eq. (2.7)) the local pitch angle $\alpha$ and parallel velocity $v_{\parallel}$ at any other point $z = z_0$ can be readily obtained from $\alpha_{\text{eq}}$ and $v_{\parallel \text{eq}}$.

$$\sin \alpha = \sqrt{\frac{B(z_0)}{B_{\text{eq}}}} \sin \alpha_{\text{eq}} \quad (3.1)$$
\[ v_{\|} = \sqrt{\frac{B(z_0)}{B_{eq}}} \frac{v_{\|\text{eq}}}{\tan \alpha_{\text{eq}}} \tan \alpha \]  

(3.2)

where \( B_{eq} \) and \( B(z_0) \) represent the equatorial and local values of the static magnetic field respectively.

In the absence of the wave a test particle described by \( \alpha_{\text{eq}0} \) and \( v_{\|\text{eq}0} \) will acquire the local pitch angle and parallel velocity as given by (3.1) and (3.2) as it moves along the field line. Significant cumulative interaction between the wave and the particle will occur only in the vicinity of the point where \( \frac{v_{\|}}{v_R} \approx 1 \), \( v_R \) being the local resonant velocity, given by \( v_R = \frac{\omega_m}{k} \). As indicated in Fig. 3.2a this condition will be satisfied in the vicinity of two locations along the field line, owing to the symmetry of the dipole field. Since the wave induced perturbations will be negligible outside these regions, the particle motion at locations other than these can still be described by (3.1) and (3.2). The encounter with the wave modifies the local \( v_{\|} \) and \( \alpha \), and through (3.1) and (3.2), the equatorial parameters \( \alpha_{\text{eq}} \) and \( v_{\|\text{eq}} \) to be associated with that particle.

**Complete interaction**

For a monochromatic CW signal traveling from south to north as shown in Fig. 3.2a, the complete interaction of each particle with the wave can be described as follows. As the particle travels south toward its point of resonance in the northern hemisphere the wave induced perturbations become more and more significant. When the particle moves past its resonance point after having been scattered in \( \alpha_{\text{eq}} \) and \( v_{\|\text{eq}} \) the perturbations get smaller until it reaches the equator. Past the
FIGURE 3.2 COMPLETE INTERACTION OF THE WAVE AND A TEST PARTICLE. (a) In general, the cyclotron resonance condition (2.30) will be satisfied at two locations N and S along the field line. (b) Adiabatic variation of the particle parallel velocity with distance from the equator. Shown in dotted lines is the resonance velocity $v_R$. The integration of the equations of motion is started at point $N'$ and ended at $S'$ for the simulation of the complete interaction.
equator, the particle approaches its second resonant point and is scattered again. Depending on the distance between the two resonance points the two encounters with the wave may or may not be in phase with each other.

To simulate this complete interaction, the equations of motion must be integrated from the time the particle is close enough (usually within 2-3% in $v_\parallel$) to its first resonance point to experience significant perturbations to the time at which the particle has passed significantly beyond its second resonance point so that the wave perturbations become insignificant. To continue the integration outside the resonance region would be useless since the changes in $\alpha_{eq}$ or $v_{\parallel eq}$ are negligible.

Figure 3.2b shows the adiabatic variation of the particle parallel velocity along the field line. Shown in dotted lines is the local resonance velocity $v_R$. The input to the main simulation part of the program is a set of values $\alpha_{eq_0}$, $v_{\parallel eq_0}$ and an initial phase $\phi_0$. With these parameters specified, the program transfers $\alpha_{eq}$ and $v_{\parallel eq}$ into local values $\alpha$ and $v_\parallel$ using (3.1) and (3.2), going progressively away from the equator, and at each point checking $|v_\parallel - v_R|$ in order to locate the vicinity of the resonance point $N$. The integration is started at a point where $v_\parallel$ is within some percentage of the resonance velocity of the northern resonance point. We have found that when the particle is more than 3% or so away from local resonance the wave induced perturbations are not significant. This value is used in our computations as indicated in Fig. 3.2b.

The motion equations are integrated using spatial steps as opposed to temporal steps. The step size used for integration is a fraction of
the step size used for the computation of the medium parameters \( \omega_H(z) \)
and \( k(z) \), so that these stored values can be used during the integra-
tion either directly or by linear interpolation between two adjacent
values. This procedure allows us to avoid the computation of the medium
and wave related quantities separately for each test particle. The in-
tegration is carried out until the time where
\[ \varepsilon = \left| \frac{V_\parallel - V_R}{V_R} \right| > 0.05 \]
and \( d\phi > 2\pi \). Both these criteria are very conservative and were established
by examining single particle trajectories. If the complete interaction
is to be simulated, the integration is carried out until the particle
passes its resonance point \( S \) in the southern hemisphere even if \( \varepsilon \)
gets to be greater than 0.05 in between its first and second resonances.

At the end of the integration the local pitch angle \( \alpha_F \) (F for
final) and parallel velocity \( V_\parallel_F \) at the point where the integration
is stopped is found. These values are then transformed through (3.1) and
(3.2) to equatorial values \( \alpha_{eq_F} \) and \( V_{eq_F} \) for that particle. The
difference \( \alpha_{eq_F} - \alpha_{eq_o} = \Delta \alpha_{eq} \) gives the total pitch angle scattering
suffered by the particle.

Figure 3.3a,b shows sample trajectories of two particles both with
\( \alpha_{eq_o} = 7^\circ \), but (a) \( V_{eq_o} \) is such that it resonates in the vicinity of
\( \pm 1^\circ \) latitude, and (b) \( V_{eq} \) is such that it resonates in the vicinity of
\( \pm 3^\circ \) latitude.

We have shown equatorial pitch angle \( \alpha_{eq} \), and the off resonance
factor \( \varepsilon \) versus time and latitude along the field line. It is clear
that as the particle approaches resonance the wave induced pitch angle
changes become more and more cumulative. Note that for \( B_w = 0 \), \( \alpha_{eq} \) is
constant with distance along the field line. As the particle moves away
FIGURE 3.3 SAMPLE PARTICLE TRAJECTORIES. Both equatorial pitch angle $\alpha_{eq}$ and off resonance factor $\varepsilon = \left| \frac{v_{\parallel}}{v_{\parallel eq_0}} - \frac{v_R}{v_R} \right|$ is shown. (a) $\alpha_{eq_0} = 7^\circ$, $v_{\parallel eq_0}$ such that it resonates at $\pm1^\circ$ latitude. (b) a particle with $\alpha_{eq_0} = 7^\circ$, $v_{\parallel eq_0}$ such that it resonates at approximately $\pm3.5^\circ$. 

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from resonance the changes in $\alpha_{eq}$ become noncumulative. Note that at
the end of the interaction the particle has attained a new pitch angle
$\alpha_{eq_f}$.

In addition to simulating the complete interaction of each test
particle with the wave, as is necessary for the full distribution compu-
tations, the computer code can also be used for studies of different por-
tions of the interaction. This is done in the next chapter where we
study the scattering of particles that are initially resonant with the
wave at the equator.
IV. STUDY OF THE INTERACTION

A. SCATTERING OF A SINGLE RESONANT SHEET

Our purpose in this section is to give a clear presentation of the different aspects of the physics of the nonlinear gyroresonance interaction.

Although the full distribution calculations give useful measurable quantities, such as the precipitated flux, they are not very helpful in understanding the physics of the interaction. The behavior of single particles and sheets provides much better insight, clarifying the effect of various parameters such as $B_w$, $\alpha_{eq}$, $n_{eq}$, L, and initial phase, $\phi_0$. We study the case of a sheet of electrons, uniformly distributed in $\phi$ and moving away from the equator. The interaction starts at the equator as depicted in Fig. 4.1. The use of initially resonant sheets enables us to present clearly the initial phase dependence. The equator is chosen because the inhomogeneity there is a minimum, thus enabling us to see the effect of the wave forces to their full extent. We must emphasize at this point that this choice is made only to simplify the presentation and not the physics. The physics, i.e. the effect and relative importance of different parameters, is unaffected. In a distribution of particles interacting with a wave at any instant of time, there will be particles with many values of $\alpha_{eq}$ and $v_{eq}$ located at all points along the field line. In our presentation of the single sheet results, we look at a sheet of electrons that meets the wave at the equator with specified $\alpha_{eq}$ and $v_{||eq}$ values. The results are qualitatively representative of those for most other sheets in the distribution, although the amount of scattering will in general be smaller for interactions.
FIGURE 4.1 DESCRIPTION OF THE INTERACTION STUDIED IN THIS SECTION. Single particles or sheets of particles that are resonant at the equator are considered.

away from the equator or for non-resonant sheets.

The parameter values used for most of our calculations are given in Table 1; they represent a realistic magnetospheric case. \( L = 4 \) is chosen because it is close to the location of the VLF transmitter at Siple, Antarctica, from which much of the experimental data have come. A wave frequency of 5 kHz is chosen, a frequency often used in the Siple wave injection experiments.

We first consider particles with an equatorial pitch angle \( \alpha_{eq_0} = 10^\circ \) and a wave amplitude of \( B_w = 10 \) mY. Figure 4.2 shows the computed trajectories for six particles distributed in initial phase \( \phi_0 \). We
FIGURE 4.2  SINGLE PARTICLE TRAJECTORIES FOR $B_0 = 10 \text{ mT}$. Both the total scattering $\Delta \alpha_{eq}$ (solid lines) and the phase $\phi$ (squared points) are shown as a function of time. All particles start at resonance at the equator (see Fig. 4.1) and move southward into the wave. Particle trajectories for 6 different initial phases are shown. The phases are chosen to illustrate typical trajectories. The dashed lines are the unperturbed phase ($\phi_0$) variation for the case of $B_w = 0$. For all trajectories $\alpha_{eq} = 10^\circ$. 

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TABLE I. PARAMETER VALUES FOR THE EXAMPLE CASE

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field line</td>
<td>$L = 4$</td>
</tr>
<tr>
<td>Equatorial gyrofrequency</td>
<td>$f_{\text{eq}} = 13.65$ kHz</td>
</tr>
<tr>
<td>Equatorial cold plasma density</td>
<td>$n_{\text{eq}} = 400$ el/cc</td>
</tr>
<tr>
<td>Wave frequency</td>
<td>$f = 5$ kHz</td>
</tr>
<tr>
<td>Equatorial parallel resonance velocity</td>
<td>$v_{\text{R}} = 1.899 \times 10^7$ m/sec</td>
</tr>
<tr>
<td>Equatorial parallel resonant energy</td>
<td>$E = 1$ keV</td>
</tr>
<tr>
<td>Refractive index at the equator</td>
<td>$n \approx 40$</td>
</tr>
<tr>
<td>Wavelength at the equator</td>
<td>$\lambda \approx 2.2$ km</td>
</tr>
</tbody>
</table>

have plotted both $\Delta \alpha_{\text{eq}}$ and the phase $\phi$ at each step of the interaction. The resonant interaction starts at the equator for all particles.

Consider for example the particle with $\phi_0 = -2\pi/3$. As this particle moves away from the equator it suffers a positive $\Delta \alpha_{\text{eq}}$, and as $\phi$ increases (the particle gyrating faster with respect to the wave) the changes in $\Delta \alpha_{\text{eq}}$ become smaller and smaller. Eventually the oscillations become insignificant and the particle leaves the interaction region with a net change in equatorial pitch angle. Note that $\alpha_{\text{eq}}$ of each of the six particles in Fig. 4.2 can be considered unaffected by the wave after about 70 msec (3° latitude). Beyond this point the wave induced particle scatterings are not cumulative.

Also shown in Fig. 4.2 is the unperturbed phase variation $\phi_u$ (i.e., the phase variation for negligible wave intensity) for some initial phases. The $\phi_u$ variation for other phases is exactly the same in form.
but shifted up or down depending on $\phi_0$ (see section 2.F).

For particles starting at resonance (i.e. $v_\parallel$ satisfies Eq. (2.30)) $\dot{\phi}_0 = 0$. For $B_w = 0$, as the particle moves away from the equator $\omega_\parallel$ increases, therefore increasing $\dot{\phi}$, which causes $\phi$ to increase as seen from the $\phi_u$ variation in Fig. 4.2. In other words, $\ddot{\phi}_0 = \ddot{\phi}_u > 0$. For $B_w \neq 0$ the interaction can be studied qualitatively as follows:

1) For negative $\phi_0$, $\ddot{v}_\parallel < 0$. (See Eq. (2.36a).) Hence $v_\parallel$ decreases while $\omega_\parallel$ increases as the particles move away from the equator. Therefore $\ddot{\phi}_0 > \ddot{\phi}_u$ and the time for which the particle stays within resonance is shortened. Also since the initial wave induced $\dot{v}_\parallel$ is negative, the local pitch angle of the particle increases, which in turn through (3.1) transforms into an increase in the equatorial pitch angle. As $\phi$ increases to the point where $\sin\phi$ changes sign (in this case becomes positive) $\ddot{v}_\parallel$ becomes positive and tends to decrease $\phi$. Since $\omega_\parallel$ keeps increasing, the wave forces offset the effects of the inhomogeneity at this stage of the interaction. However, there already is a large $\dot{\phi}$ and the particle is no longer near resonance. Therefore if the wave amplitude is not strong enough to cause an oscillation (or reversal) in phase (i.e. trapping) the phase angle $\phi$ continues to increase. When $\phi$ comes to the point where $\sin\phi$ again changes sign, $\ddot{v}_\parallel$ again becomes negative. The periodic changes of sign of $\ddot{v}_\parallel$ lead to oscillations in $\Delta\alpha_{eq}$. This behavior is most clearly illustrated by the case of larger wave amplitudes, as in Fig. 4.3. Since $\omega_\parallel$ continuously increases as the particle moves away from the equator the period of these oscillations decreases as the particle moves away from resonance. Eventually the particle acquires a net pitch angle change $\Delta\alpha_{eq}$. Typical trajectories
for the negative initial phase case are shown in Fig. 4.2.

2) For positive $\phi_0$, $\dot{\phi}_0 > 0$, and the wave forces offset the effects of the inhomogeneity. Therefore $\ddot{\phi} < \phi_u$ and the time during which particle stays within resonance is increased, hence allowing more scattering due to more exposure to cumulative interaction. For the case of $B_w = 10 \ m_7$, given in Fig. 4.2, we see that final scatterings for $\phi_0 > 0$ are generally larger than those for $\phi_0 < 0$, although the final scattering depends, as we shall later see, on a wide variety of parameters. Note in Fig. 4.2 that although the initial $\dot{\phi}_0$ is a maximum for $\phi_0 = \pi/2$, the final scattering is the largest for $\phi_0 = 2\pi/3$. This occurs because for $\phi_0 = 2\pi/3$ the wave and inhomogeneity forces balance each other over a longer distance and hence $\dot{\phi}$ stays very close to zero, therefore increasing the time spent within resonance.

For an initial pitch angle of $10^\circ$, a wave amplitude of $10 \ m_7$ is not large enough to cause more than one oscillation in phase $\phi$. That is, it cannot trap the particles for long in the wave's potential well. In Fig. 4.3 we give the trajectories of seven particles for $B_w = 50 \ m_7$.

In comparing Figs. 4.2 and 4.3 we make the following observations:

1) The oscillations in $\Delta x_{eq}$ as the particle moves away from resonance are larger in amplitude for $B_w = 50 \ m_7$ than for $B_w = 10 \ m_7$. This is expected since the scattering at each point is proportional to $B_w$.

2) For some of the phases ($\phi_0 = -\pi/2, -\pi/3, 0, \pi/3$) the particle phase $\phi(t)$ makes more than one oscillation. Hence the particle is phase trapped, although only for a short while. The most strongly trapped particle is the one with $\phi_0 = 0$. 

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FIGURE 4.3 SINGLE PARTICLE TRAJECTORIES FOR $B_0 = 50$ mV.
The format is the same as Fig. 4.2.
For all trajectories, $\Delta a_{eq} = 10^\circ$. 
3) All the trapped particles end up with a negative $\Delta \alpha_{eq}$. This is because for trapped particles $v_{||} \approx v_R = \frac{\omega_H - \omega}{k}$ and $\dot{v}_{||} \approx \dot{v}_R$. Since $\dot{v}_R > 0$ when moving southward from the equator, $\dot{v}_{||} > 0$ and, from Eq. (2.36b), $\dot{v}_\perp < 0$. Thus the pitch angle must decrease for all trapped particles as they move away from the equator.

4) Consider the trajectory for the $\phi_0 = -\pi/3$ particle. The second minimum of $\Delta \alpha_{eq}$ (at $t = 43$ ms) seems much sharper than for other cases. Note that the particles shown in Fig. 4.3 have an equatorial pitch angle of $\alpha_{eq} = 10^\circ$. The second minimum in $\Delta \alpha_{eq}$ for $\phi_0 = -\pi/3$ is at about $\Delta \alpha_{eq} = 9.5^\circ$. In other words, as the pitch angle continuously decreases it has reached a point where it is very close to zero. At those low pitch angles $v_\perp$ is very small and therefore the wave term ($\sim \cos \phi \frac{\cos \phi}{v_\perp}$) in Eq. (2.36c) becomes significant, causing a large change in phase which prevents the pitch angle from reaching zero. This effect is called the 'loss cone reflection' effect and is discussed in Appendix B. This example shows the importance of not deleting this term in the computations. Without this term the pitch angle would have gone negative. At the third minimum of $\Delta \alpha_{eq}$ for $\phi_0 = 0$ we have the same effect, again clearly seen by an abrupt change in $\phi$.

After this discussion of the single particle trajectories, it is further enlightening to study the collective motions of sheets of particles. As our tool for this study we consider the variation of the final scattering $\Delta \alpha_{eq}$ versus initial phase. We have found that the nature of the interaction and the effects of different parameters are most clearly presented in such a format. Below, we give results of our computations for a sheet of 24 electrons equally distributed in initial
phase $\phi_0$. We consider different cases to isolate the effects of parameters such as $B_w$, $\alpha_{eq}$, $n_{eq}$, position of resonance, etc. The results are obtained by integrating the full equations of motion for each of the 24 electrons in the sheet to obtain the total final scattering $\Delta \alpha_{eq}$. We then plot $\Delta \alpha_{eq}$ vs. initial phase $\phi_0$, and discuss each result qualitatively.

a) Effect of the Wave Amplitude

Figure 4.4a shows $\Delta \alpha_{eq}$ vs. $\phi_0$ for the parameters as given in Table 1 and for $\alpha_{eq0} = 10^\circ$. We have given results for a range of wave amplitudes from $B_w = 1 \text{ mY}$ to $B_w = 50 \text{ mY}$. For low wave amplitudes such as $B_w = 1 \text{ mY}$ and $3 \text{ mY}$ we have 'linear' scattering. In other words, for these cases the inhomogeneity is the dominant factor in controlling $\phi(t)$ rather than the wave forces. Therefore $\phi(t) = \phi_u(t)$ and linear theory can be applied. That the shape of the $\Delta \alpha_{eq}$ vs. $\phi_0$ curves is approximately sinusoidal for these cases is shown in Appendix A. In the rest of this report the interaction will be termed to be in the 'linear' mode whenever $\Delta \alpha_{eq}$ vs. $\phi_0$ is approximately sinusoidal.

For $B_w > 7 \text{ mY}$ the curves start to change shape. For $B_w = 50 \text{ mY}$ we have what we call the 'trapped' mode. The trapped mode is one in which the wave forces play the dominant role in controlling $\phi$. The phase $\phi$ in this case goes through more than one oscillation due to sign changes of $\dot{\phi}$. As discussed above in connection with Fig. 4.3 the trapped particles end up with a net negative change in pitch angle, thus producing a $\Delta \alpha_{eq}$ vs. $\phi_0$ variation as shown in Fig. 4.4a for the $B_w = 50 \text{ mY}$ case. Note that only particles in a range around $\phi_0 = 0$ are trapped, the initial phases of other particles are not appropriate.
FIGURE 4.4  TOTAL SCATTERING, $\Delta \alpha_{eq}$, VERSUS INITIAL PHASE FOR DIFFERENT WAVE AMPLITUDES, FOR $L = 4$ AND $n_{eq} = 400$ el/cc. (a) $\alpha_{eq} = 10^\circ$. (b) $\alpha_{eq} = 30^\circ$. Each square in the figures shows the scattering of an individual test $q_{eq}$ particle. 24 particles, uniformly distributed in $q_{eq}$, are used in each sheet.
for trapping. This is also seen in Fig. 4.3.

Figure 4.4 gives only the final net scatterings. No time parameter is involved. For each case, the equations have been integrated for each electron in the sheet until the particle is no longer significantly affected by the wave. The total pitch angle change $\Delta \alpha_{eq}$ is then plotted against $\phi_0$. Note that for $7 \text{ mY} < B_w < 50 \text{ mY}$ the $\Delta \alpha$ vs $\phi_0$ curves resemble neither the linear nor the trapped mode. This is because for these transition values neither the wave force nor the inhomogeneity is clearly dominant. For $B_w = 3 \text{ mY}$ and $5 \text{ mY}$ we observe from Fig. 4.4a that there is an asymmetry between scattering for negative and positive initial phases. This comes about because, although for all $\phi_0$ the inhomogeneity is the dominant factor, for positive $\phi_0$ the wave forces offset the effects of the inhomogeneity whereas for negative $\phi_0$ the wave force adds to the effect of the inhomogeneity. Therefore for positive $\phi_0$ the particle stays in resonance for a longer time and hence experiences larger scattering.

For $B_w = 50 \text{ mY}$ Fig. 4.4 shows that the maximum negative scattering for $\alpha_{eq} = 10^\circ$ is about $\Delta \alpha_{eq} = 9^\circ$. Thus the absolute pitch angle for these particles is reduced nearly to zero. Similarly for $\alpha_{eq} = 30^\circ$ the maximum negative scattering is $24^\circ$, bringing this particle to the edge of the loss cone.

One important point that must be kept in mind is the following: Figure 4.4a shows results for a single sheet traveling away from the equator. Consider the case of a sheet traveling towards the equator.
By studying Eq. (2.36c) we see that \( \Delta \alpha_{eq} \) vs. \( \phi_0 \) variations would be approximately a mirror image of the result for the \( B_w = 50 \text{ m}\gamma \), with the trapped particles having a net positive pitch angle change. Therefore one should not try to make predictions about the precipitated flux or other full distribution quantities using only these results. The single sheet results given in this section are intended only as an aid to understanding the interaction, and sorting out the dependence of \( \Delta \alpha_{eq} \) on various parameters.

For higher \( \alpha_{eq} \), one would expect the deviation from the linear mode to start at lower amplitudes. In Fig. 4.4b we show the results for \( \alpha_{eq} = 30^\circ \) in the same format as Fig. 4.4a. We see that the case of \( B_w = 3 \text{ m}\gamma \) is as much away from the linear mode as the case of \( B_w = 10 \text{ m}\gamma \) for \( \alpha_{eq} = 10^\circ \). This is explained by the fact that the wave induced \( \dot{v}_\parallel \) is proportional to \( v_\perp B_w \), and therefore to \( \tan \alpha_{eq} \). As \( B_w \) is increased further we again have a similar kind of transition to the trapped mode. However the minima of \( \Delta \alpha_{eq} \) are much deeper than the \( \alpha_{eq} = 10^\circ \) case. For \( B_w = 50 \text{ m}\gamma \), for example, the most stably trapped particle (\( \phi_0 = 0 \)) undergoes a net scattering of almost \( 24^\circ \). The behavior in Figs. 4.2, 4.3 and 4.4a,b also clearly shows the difference between coherent and incoherent interactions. In the coherent interaction the particle can be phase locked with the wave and lost almost all of its perpendicular energy in a single encounter with the wave. On the contrary, in an incoherent interaction, the scatterings suffered by the particle at each instant are random in direction and magnitude. As a result the particles execute a random walk in pitch angle space and a diffusion in the average direction of the field results. Therefore the
net total scatterings at each encounter with the wave are generally much smaller.

b) **Comparison with the Linear Theory**

At this point we are ready to compare quantitatively the results of our full analysis with those of linear theory. For purposes of comparison we have arranged our computer program to integrate the motion equations using linear theory if desired. Figure 4.5 shows a comparison of the linear and nonlinear analyses for $\alpha_{\text{eq}} = 10^\circ$. For the linear case the $\Delta \alpha_{\text{eq}}$ vs. $\phi_0$ variation is proportional to $B_w \sin(\phi_0 + \theta)$, where $\theta$ is defined in Appendix A. We see from Fig. 4.5 that the difference between the linear and the nonlinear results becomes apparent for $B_w \gtrsim 5 \, m_\gamma$.

Even for $B_w = 3 \, m_\gamma$ the asymmetry between the scatterings for positive and negative $\phi_0$ is apparent for the nonlinear case. Note that for higher $\alpha_{\text{eq}}$ the deviation from linear theory occurs at lower wave intensities. For instance, for $\alpha_{\text{eq}} = 30^\circ$ and $B_w = 3 \, m_\gamma$ the interaction is clearly nonlinear as can be seen from Fig. 4.4b. Note also that for the linear case the phase variation is the same for all 24 particles in the sheet. Therefore they all spend the same amount of time in resonance.

Figure 4.6a,b compares the linear and nonlinear root mean square scattering (i.e. $\sqrt{<\Delta \alpha^2>}$ where $<>$ denotes averaging over the initial phases) and the mean value (i.e. $<\Delta \alpha>$) of the $\Delta \alpha_{\text{eq}}$ vs. $\phi_0$ curves for both $\alpha_{\text{eq}} = 10^\circ$ and $30^\circ$ cases. We see that as the wave amplitude is increased there is an increasing mean value for the nonlinear cases whereas the mean value for the linear case is zero for all wave amplitudes.

For $\alpha_{\text{eq}} = 10^\circ$ and for small wave fields of up to $5 \, m_\gamma$ the rms scattering for both cases is about the same. However as the wave amplitude
FIGURE 4.5  COMPARISON OF RESULTS OF LINEAR THEORY AND THE FULL NON-LINEAR ANALYSIS. The total scattering vs. $\phi_0$ is given for $\alpha_{eq} = 10^\circ$ and for different wave amplitudes. The format is very similar to that of Fig. 4.4.
FIGURE 4.6  RMS AND MEAN SCATTERING FOR LINEAR AND NONLINEAR ANALYSES AS A FUNCTION OF WAVE AMPLITUDE. The mean scatterings for the linear case is zero. (a) $\alpha_{eq} = 10^\circ$. (b) $\alpha_{eq} = 30^\circ$.
is increased the rms scattering for the nonlinear case deviates from the linear case in which the rms scattering is directly proportional to $B_w$. Therefore the use of linear theory when $B_w > 3 m_r$ causes an error in both mean and rms scatterings.

Since for $B_w > 40 m_r$, $\Delta x_{eq} = x_{eq}$, the rms scattering saturates for the $x_{eq} = 10^\circ$ case; however for the higher initial pitch angle ($x_{eq} = 30^\circ$) case we see that the rms scattering is considerably larger than the linear one over a wide range of wave amplitudes. The scattering is larger because the nonlinear formulation takes particle trapping into account and this effect considerably increases the length of the interaction region over which scattering takes place.

Since the $\Delta x_{eq}$ vs $\phi_0$ curve for the linear case is found to have a sinusoidal shape (see Appendix A) it is possible to readily identify linear or trapped behavior by qualitatively examining the $\Delta x_{eq}$ vs $\phi_0$ curves. For example, for the case of Fig. 4.4a we see that deviation from a sinusoidal shape starts around $B_w = 5 m_r$ and hence linear theory should be applicable only for $B_w < 5 m_r$. In the following cases we shall use this concept to study our results.

c) Dependence on Equatorial Pitch Angle

To isolate the effect of the initial equatorial pitch angle we hold the wave amplitude constant at $B_w = 10 m_r$ and for the parameters of Table 1 compute the scatterings for different $x_{eq}$. Figure 4.7 shows $\Delta x_{eq}$ vs $\phi_0$ curves parameterized in $x_{eq}$. A wide range of pitch angles $3^\circ \leq x_{eq} \leq 85^\circ$ is covered. Since the wave induced $v_\parallel$ is proportional to $v_\parallel$ and hence to $\tan x_{eq}$, for small enough pitch angles a linear $\Delta x_{eq}$ vs $\phi_0$ variation is expected. Indeed from Fig. 4.7 we
FIGURE 4.7 INITIAL PITCH ANGLE DEPENDENCE OF THE TOTAL SCATTERING VS. $\phi_0$ FOR $n_{eq} = 400 \text{ e1/cc AND } B_w = 10 \text{ mV}$. The format is the same as that of Fig. 4.5.
see that the variation for $\alpha_{eq} = 3^\circ$ is essentially linear. The deviation from linearity increases as $\alpha_{eq}$ is increased and we have the trapped mode for $\alpha_{eq} > 40^\circ$. The transition from the linear to the trapped mode is similar to that shown in Fig. 4.4a.

For $\alpha_{eq} > 60^\circ$, the minimum of the $\Delta \alpha_{eq}$ vs. $\phi_0$ curves starts to contract. Although the shape of $\Delta \alpha_{eq}$ vs. $\phi_0$ stays in the trapped mode, the total scattering for each particle continuously decreases.

The reason for this is that for these high values of $\alpha_{eq}$ the variation in the particles' $v_\parallel$ and $v_\perp$ due to the adiabatic mirror forces (second terms on the r.h.s. in Eqs. (2.36a,b) becomes more and more rapid, therefore decreasing the time spent in resonance and resulting in smaller scattering.

d) **Dependence on the Equatorial Cold Plasma Density**

One other parameter which defines the properties of the medium and the wave is $n_{eq}$, the equatorial cold plasma density. The wave number $k$ is proportional to $n_{eq}$ (see (2.25)). In Fig. 4.8 we give results for a $B_w = 50$ mT wave and $\alpha_{eq} = 10^\circ$ particles at $L = 4$. We show $\Delta \alpha_{eq}$ vs. $\phi_0$ curves for different $n_{eq}$ values, ranging from 400 el/cc (inside the plasmapause) to 1 el/cc (outside the plasmapause).

Figure 4.8 shows that for $n_{eq} = 400$ el/cc a trapped mode exists. Decreasing $n_{eq}$ results in a transition similar to those seen before, to a linear mode for $n_{eq} = 1$ el/cc. This result shows clearly that interactions outside the plasmapause result in less scattering than those inside the plasmapause. This interesting result can be understood as follows: As $n_{eq}$ decreases the wave phase velocity increases and the parallel resonant velocity becomes higher. Higher energy particles move
FIGURE 4.8 VARIATION OF TOTAL SCATTERING WITH EQUATORIAL COLD PLASMA DENSITY, $n_{eq}$. For all cases $L = 4$, $f = 5$ kHz, $B_z = 50$ mT, and $\alpha_{eq} = 10^\circ$. Note that for $n_{eq} = 1$ el/cc the parallel resonant particle velocity is in the relativistic range. We have not accounted for the relativistic correction in our calculation. However this correction will only vary the amplitude of the scattering. Our main concern in this section is the shape of the $\Delta \alpha_{eq}$ vs. $\phi_0$ curves which will not be affected by that correction.
more quickly through the wave and hence have less time in which to be scattered. Note that although the equatorial parallel resonant energy is about 1 keV for \( n_{eq} = 400 \text{ el/cc} \) it is approximately 40 keV for \( n_{eq} = 10 \text{ el/cc} \).

e) *Dependence on the Geomagnetic Latitude Around Which the Resonance Occurs*

Figure 4.9 shows \( \Delta \alpha_{eq} \) vs. \( \phi_0 \) for \( B_w = 50 \text{ mT} \) and \( \alpha_{eq} = 10^\circ \). The curves in this case are parametric in the location of resonance. Hence the first sheet starts from the equator at resonance, the next starts at resonance at 2° latitude south of the equator and the last sheet starts at resonance at 10° latitude. All sheets travel southward away from the equator. We observe that for resonance at the equator we have a trapped mode, but as the resonance latitude is increased to 10° there is a transition to the linear mode. This is expected since the gradient of the earth's magnetic field increases with distance from the equator and for high enough latitudes the effects of inhomogeneity can no longer be offset by the wave forces. Because the inhomogeneity becomes the controlling factor at higher latitudes the shape of \( \Delta \alpha_{eq} \) vs. \( \phi_0 \) returns to the linear form. Note that this behavior is limited to constant frequency waves. For waves of changing frequency, maximum interaction may occur off the equator, as suggested by Helliwell [1967, 1970].

B. **QUANTITATIVE CRITERIA FOR DETERMINING THE APPLICABILITY OF THE LINEAR THEORY**

The linear theory procedure for computing the wave induced scatterings \( v_\parallel \) and \( v_\perp \) was described in section 2.F. A unique feature of
FIGURE 4.9  TOTAL SCATTERING AS A FUNCTION RESONANCE POINT ALONG THE FIELD LINE. For all cases, $L = 4$, $f = 5$ kHz, $B_w = 50$ mG, $\alpha_{eq} = 10^\circ$ and $n_{eq} = 400$ el/cc.
linear theory is that the path length over which resonance occurs is independent of wave amplitude. For instance if an electron is at resonance ($\phi=0$) with the wave at some point $z_1$ along the magnetic field line, linear theory predicts that resonance will effectively terminate at a point $z_2$, defined implicitly by the approximate relation

$$\Delta \phi = 1 \text{ radian} = \int_{z_1}^{z_2} \frac{dz}{\frac{\omega}{v_R} - \omega - kv_y}$$

(4.1)

where $\omega$ is the electron gyrofrequency at the resonance point $z_1$, and $v_R = \frac{\omega_B - \omega}{k}$. It can be seen that $z_2$ does not depend on $B_w$ since in linear theory the wave induced changes in $v_w$ are ignored in integrating Eq. (2.36c). A rough method of establishing the validity of linear theory in any particular case involves the comparison of the time the particle spends in the linear resonance region $t_R = \frac{z_2 - z_1}{v_R}$, with the trapping time of the particle in the potential well of the wave,

$$t_T = 2\pi \left[ \frac{eB}{m} v_y \right]^{-1/2}.$$  When $t_{R_T}^{-1} << 1$, it can be concluded that nonlinear effects will be negligible and that linear theory can be used, while if $t_{R_T}^{-1} > 1$, it can be concluded that nonlinear effects must be included. The major problem with this criterion is that it cannot be clearly applied when $t_{R_T}^{-1} = 1$, since the actual nonlinear resonance time will generally exceed $t_R$. In this case it is of interest to use our results to attempt to establish a more accurate method to determine the applicability of the linear theory.

The very similar behavior of the $\Delta \alpha_{eq}$ vs. $\phi_0$ curves as displayed in Figs. 4.4 through 4.9 suggests that the nature of the interaction may be described by a single parameter which is a function of $B_w$, $\alpha_{eq}$, $\eta_{eq}$
(or $k$) and $\frac{\partial \omega_H}{\partial z}$. The controlling factor in the interaction is the variation of phase, $\phi$, as given by Eq. (2.36c). Note that for all sheets considered in Figs. 4.4 through 4.9 the set of values used for $\phi_0$ is the same regardless of the values of the different parameters. For resonant particles $\ddot{\phi}_0 = 0$, for nonresonant particles $\ddot{\phi}_0$ is a non-zero constant. The different behavior of the sheets, whether they interact in the trapped or linear mode, is determined by the initial rate of change of $\dot{\phi}$, i.e. $\ddot{\phi}_0$. Neglecting the wave term in Eq. (2.36c) (the term proportional to $\frac{\cos \phi}{v}$) $\ddot{\phi}$ is given by Eq. (2.25). Rewriting that we have

$$
\dot{\phi} = \left[ \frac{3}{2} \frac{v_\perp^2}{v_H} + k \frac{v_\perp^2}{2 \omega_H} \right] \frac{\partial \omega_H}{\partial z} - k \left( \frac{eB}{m} \right) v_\perp \sin \phi
$$

(4.2)

The first and second terms on the r.h.s. in Eq. (4.2) represent the inhomogeneity and wave forces respectively. We now define a quantity $\rho$:

$$
\rho = \frac{\frac{eB}{m} v_\perp}{\left[ \frac{3}{2} \frac{v_\perp^2}{v_H} + k \frac{v_\perp^2}{2 \omega_H} \right] \frac{\partial \omega_H}{\partial z}}
$$

(4.3)

This quantity is the ratio of the maximum absolute values of the wave and inhomogeneity terms in Eq. (4.2). Hence the value of $\rho$ is an indication of the relative effectiveness of these terms. Similar analyses to study the relative effects of the wave and inhomogeneity terms were used by previous authors [Dysthe, 1971; Nunn, 1974; Karpman, 1974; Roux and Pellat, 1976]. Here we carry this work a step forward:

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We can rewrite $\rho$ to obtain:

$$
\rho = \frac{2k \left( \frac{eB_W}{m} \right) \tan \alpha}{3 + \left( \frac{\omega_H - \omega}{\omega_H} \right) \tan^2 \alpha} \frac{3\omega_H}{3z} \tag{4.4}
$$

where we have used (2.30) to eliminate $k$ in the denominator. Note that $\rho$ is a dimensionless quantity dependent on $B_w$, $\alpha$(local pitch angle), $k$ and $\frac{3\omega_H}{3z}$.

We expect that $\rho$ will be a useful quantity for differentiating between linear and nonlinear interactions. For instance linear theory predicts that for a particle that is resonant with the wave at some point $z_1$ along the magnetic field line the resonance will effectively terminate at a point $z_2$ defined by (4.1). If, for any particular case, we compute the quantity $\rho$ at $z_2$ we can determine whether the resonance interaction will proceed significantly beyond $z_2$. Thus if $\rho < 1$ at $z_2$ we can conclude that the wave forces are weaker than the inhomogeneity and that linear theory should apply. If $\rho > 1$ at $z_2$ we can conclude that the resonance interaction will proceed significantly beyond $z_2$, that linear theory is not appropriate, and that a full nonlinear treatment should be employed.

In practice the evaluation of $\rho$ is simplified by the fact that for the dipole model $\rho(z_1) = \rho(z_2)$, and thus $\rho$ can generally be simply evaluated completely in terms of the initial conditions. A slight complication occurs when the initial resonance point approaches the magnetic equator, since then $\rho(z_1) \to \infty$ as $\frac{3\omega_H}{3z} \to 0$. In this case we must evaluate $\rho$ at the point $z_2$. An analytic expression for $z_2$ can be found.
using (4.1) and (4.2) in the limit \( B_w = 0 \). From Eq. (4.2) we have

\[
\phi = \left[ \frac{3}{2} \frac{k \nu^2}{2 \omega_w} \right] \frac{\partial \omega_H}{\partial z} \tag{4.5}
\]

or

\[
\frac{d^2 \phi}{dz^2} = \frac{1}{2} \frac{\nu_H}{\nu} \left[ 3 + \left( \frac{\omega_H - \omega}{\omega_H} \right) \tan^2 \alpha \right] \frac{\partial \omega_H}{\partial z} \tag{4.6}
\]

where we have used (2.30) to eliminate \( k \). In a dipole field, for locations close to the equator the variation of the gyrofrequency is very closely approximated by [Helliwell, 1970].

\[
\omega_H = \omega_{H_0} \left[ 1 + \frac{q}{Z} \left( \frac{z}{R} \right)^2 \right] \tag{4.7}
\]

where \( \omega_{H_0} \) is the angular electron gyrofrequency at the equatorial plane, \( R \) is the geocentric distance and \( z \) is the distance from the equator measured along the field line. Equation (4.6) can be twice integrated from \( z = z_1 = 0 \) to some point \( z \) by assuming that \( v = v_R \) and \( \omega_H = \omega_{H_0} \) and \( \alpha = \alpha_0 \) are constant. We obtain

\[
\phi = \frac{\sigma}{v_R} \frac{\omega_{H_0}}{2} \frac{3}{2} \frac{z^3}{R^2} + \phi_0 \tag{4.8}
\]

where \( \sigma = \frac{1}{2} \left[ 3 + \frac{\omega_H - \omega}{\omega_{H_0}} \tan^2 \alpha_0 \right] \) and \( \phi_0 \) is the initial phase. From (4.1) point \( z_2 \) is defined as that point at which \( \Delta \phi = 1 \) radian.

Therefore

\[
z_2 = \left[ \frac{2 v_R R^2}{3 \sigma \omega_{H_0}} \right]^{1/3} \tag{4.9}
\]
Therefore for resonances at the equator \( \rho \) can be evaluated at point \( z_2 \) given by (4.9) using the value \( \left. \frac{3\omega_H}{z} \right|_{z_2} \approx 9\omega_H \left( \frac{z_2}{R^2} \right)^{1/3} \). For the parameters of Table 1,

\[
z_2 = 576[3 + 0.63 \tan^2 \alpha_o]^{-1/3} \text{ km.} \tag{4.10}
\]

and using these we obtain

\[
\left. \frac{3\omega_H}{z} \right|_{z_2} = 6.86 \times 10^{-1} \left[3 + 0.63 \tan^2 \alpha_o \right]^{1/3} \text{ rad/km.} \tag{4.11}
\]

The quantity \( \rho(z_2) \) can be determined simply in terms of initial values by making use of the fact that \( \omega_H(z_2) = \omega_H \) and \( \alpha(z_2) = \alpha_o \). Thus two important features of the quantity \( \rho \) are: 1) its value can be used to determine when linear theory is appropriate, and 2) it can be evaluated simply in terms of the initial conditions alone.

In the following we demonstrate that the transitions from the linear to trapped mode in Figs. 4.4 through 4.9 occur at or near the \( \rho = 1 \) threshold.

In Table 2 we give the values of \( \rho \) for the cases covered in Figs. 4.4 through 4.9. First consider the wave amplitude dependence as illustrated in Fig. 4.4a for \( \alpha_{eq_o} = 10^\circ \). Table 2a gives \( \rho \) for different \( B_w \). From Figs. 4.4a we see that significant deviations of the \( \Delta\alpha_{eq} \) vs \( \phi_o \) curves from a linear, near-sinusoidal, variation start at about \( B_w = 7 \text{ m}_\gamma \). Since \( \rho = 0.9 \) for 7 \text{ m}_\gamma we can loosely conclude that linear theory fails for \( \rho > 1 \).

Table 2b shows the corresponding values of \( \rho \) for the different wave amplitudes of Fig. 4.4b, for which \( \alpha_{eq_o} = 30^\circ \). From that figure it is evident that a large deviation from the linear mode is seen for...
TABLE II. VALUES OF $\rho$ FOR DIFFERENT CASES

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha_{eq}$</th>
<th>$L$</th>
<th>$n_{eq}$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$10^\circ$</td>
<td>4</td>
<td>400 e1/cc.</td>
<td>0.1</td>
</tr>
<tr>
<td>$B_w (\text{mY})$</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>30</td>
<td>50</td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td>1.3</td>
<td>3.8</td>
<td>6.4</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>$30^\circ$</td>
<td>4</td>
<td>400 e1/cc.</td>
<td>0.4</td>
</tr>
<tr>
<td>$B_w (\text{mY})$</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>30</td>
<td>50</td>
<td>2.8</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>12.0</td>
<td>20.0</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>$10^\circ$, $B_w = 50 \text{ mY}$</td>
<td>4</td>
<td>400 e1/cc.</td>
<td>6.4</td>
</tr>
<tr>
<td>$n_{eq}(\text{e1/cc})$</td>
<td>400</td>
<td>200</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
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<td>1</td>
<td>4.5</td>
</tr>
<tr>
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<td>2.3</td>
<td>1.0</td>
<td>0.7</td>
<td>2.3</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>$B_w = 10 \text{ mY}$, $L = 4$, $n_{eq} = 400 \text{ e1/cc}$</td>
<td>1</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>$\alpha_{eq}(\text{deg})$</td>
<td>30</td>
<td>70</td>
<td>85</td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>10.6</td>
<td>8.9</td>
<td>1.3</td>
</tr>
</tbody>
</table>

$B > 3 \text{ mY}$ compared to an almost linear result for $B_w = 1 \text{ mY}$. Hence we can say that the deviation from linear theory starts at $B_w = 2-3 \text{ mY}$. We see from Table 2b that $\rho = 1$ for these values. Therefore again $\rho = 1$ is a reasonably good indication of the point beyond which linear theory becomes inaccurate.

The value of $\rho$ for the different $n_{eq}$ values of Fig. 4.8 is given in Table 2c. We see from Fig. 4.8 that linear theory applies for
$n_{eq} \leq 10 \text{ el/cc}$. Since $\rho = 1.0$ for $n_{eq} = 10 \text{ el/cc}$ we again see that linear theory threshold is at $\rho = 1$.

Figure 4.9 showed $\Delta \alpha_{eq}$ vs. $\phi_0$ for different resonance points along the field line. It is evident from the figure that we have a well-defined linear mode for resonances at latitudes $\geq 10^\circ$. For particles resonant at $10^\circ$, $\rho(z_1) \approx \rho(z_2)$ and we can use $\frac{\partial \omega}{\partial z}$ at $10^\circ$ latitude for computing $\rho$. This gives $\rho = 0.71$. Note that the $\Delta \alpha_{eq}$ vs. $\phi_0$ curve for $10^\circ$ latitude is very closely linear, whereas the one for $5^\circ$ latitude is different from a linear mode. Here we have $\rho = 1.7$ for resonance at $5^\circ$ and $\rho = 1.0$ at $7.5^\circ$ latitude. Therefore we see that $\rho = 1$ is again a valid threshold point for determining linearity.

As a final test of our criterion we use the "experimental data" given in Fig. 4.7. Table 2d gives values of $\rho$ for the $\alpha_{eq}$ values used in that figure. The deviation from the linear mode starts around $\alpha_{eq} = 7^\circ$ (which is not shown in Fig. 4.7) and $\rho = 0.9$ at that point, again very close to the $\rho = 1$ criteria which was established above. We would like to note here that the sharpness of the minima for $\alpha_{eq} = 5^\circ$ given in Fig. 4.7 is not so much due to the deviation from linearity as it is due to the wave term in Eq. (2.36c) since at that point $\Delta \alpha_{eq} = \alpha_{eq}$.

As we examine Table 2d we see that $\rho$ reaches a maximum at about $\alpha_{eq} = 50^\circ$. The pitch angle dependence of $\rho$ is given from Eqs. (4.4) and (4.10) as:

$$\rho \propto \frac{\tan \alpha}{\frac{\omega - \omega_H}{\omega_H} \tan \alpha}^{2/3}$$

(4.12)
The maximum of this function occurs when \( \left( \frac{\omega H}{\omega_0} \right) \tan^2 \alpha = 9 \). For the parameters of Table 1, this maximum is at \( \alpha = 75^\circ \). From Fig. 4.7 we see that for \( \alpha_{eq} > 60^\circ \) the \( \Delta\alpha_{eq} \) vs. \( \phi_0 \) curves start to shrink while still staying in the trapped mode. This is readily explained by the fact that when the adiabatic accelerations \( \dot{v}_\parallel \) and \( \dot{v}_\perp \) are large the resonance time of the particle is decreased and this results in smaller scattering.

The correlation between Table 2d and Fig. 4.7, i.e., the fact that both \( \rho \) and the scattering curves reach a maximum for \( \alpha = 65^\circ - 75^\circ \) is very interesting. It indicates that not only is \( \rho \) useful as a threshold criteria for determining linearity, but also its absolute magnitude may be a direct indication of the amount of scattering. We have studied this question extensively and have determined that \( \rho \) can be successfully used as an empirical parameter for easy computation of the scattering coefficients for the trapped mode. This result is discussed in the next section, in connection with Fig. 4.19.

With the above results and comparisons we conclude the following:

1) The quantity \( \rho \) can be used to determine whether or not the linear theory is applicable for given interaction parameters.

2) For \( \rho \ll 1 \) (\( \rho < 0.7 \)) linear theory results are close to those of the full nonlinear analysis.

3) For \( \rho \gg 1 \) (\( \rho > 3 \)) we have a trapped mode and the linear theory is not applicable.

4) The linear theory results begin to deviate significantly from those of the full nonlinear analysis for \( \rho > 1 \).
Therefore we have a convenient and simple criteria for determining the applicability of the linear theory. The procedure is to compute $\rho$ for the parameters of the problem at hand and decide according to the criteria cited above.

The criterion we have established above uses only the initial values of the parameters and variables. Therefore one need not compute the particle trajectories in order to use this criterion. This fact makes the method very useful in determining whether or not one can use a simple linear analysis for an interaction of a particle with a given pitch angle at any point along the field line with a wave of any frequency and wave amplitude.

Previous authors, e.g., Ashour-Abdalla [1972], have used linear theory without quantitative justification. For the parameters of Ashour-Abdalla [1972] and for equatorial interactions we compute $\rho = 0.6$ for $\alpha_{eq} = 10^\circ$, $\rho = 1.7$ for $\alpha_{eq} = 30^\circ$ and $\rho = 2.5$ for $\alpha_{eq} = 50^\circ$. Note that none of these values for $\rho$ satisfy $\rho \ll 1$ which guarantees safe application of linear theory. Furthermore for $\alpha_{eq} > 30^\circ$, $\rho > 1$. For this reason we believe that the scattering coefficients given in that paper for resonances close to the equator (within 500-1000 km) are inaccurate for high pitch angles.

C. THE TRAPPED MODE

In this section we extend our study of the scattering of an initially resonant sheet starting at the equator and moving southward as depicted in Fig. 4.1. Here we consider the scattering for higher wave intensities and higher pitch angles emphasizing the cases for which $\rho > 1$ so that the interaction is in the trapped mode. The $\Delta \alpha_{eq}$ vs $\phi_0$ curves for
this case have the shape shown in Fig. 4.10 for $\alpha_{eq} = 70^\circ$ and various values of $B_w$. As seen from this figure and Fig. 4.4 once the trapped mode is reached, further increase in $B_w$ results in more scattering for all the trapped particles, while the general shape of the $\Delta \alpha_{eq}$ vs. $\phi_0$ curve stays the same. Note that trapped particles are those for which $\phi$ makes more than one oscillation. For this case of poleward motion the trapped particles all suffer a decrease in pitch angle as described in section 2.A in connection with Figs. 4.3 and 4.4.

Owing to the characteristic shape of the $\Delta \alpha_{eq}$ vs. $\phi_0$ curves it is conceivable that the behavior of the sheet could be studied by considering only one of the trapped particles, possibly the particle which is the most stably trapped, i.e. that for which $\phi_0 = 0$. To investigate this possibility we have plotted in Fig. 4.11 the total scattering $\Delta \alpha_{eq}$ for the $\phi_0 = 0$ particle versus the rms scattering $\sqrt{<\Delta^2>}$ where $<>$ denotes averaging over all initial phases. We have shown results for initial pitch angles from $10^\circ$ to $70^\circ$. For each pitch angle $\Delta \alpha_{eq}$ is varied by changing the wave intensity. We see from Fig. 4.11 that, within an error of less than 10%, the $\Delta \alpha_{eq}$ for $\phi_0 = 0^\circ$ particle and the rms scattering are linearly related with a proportionality constant of $\sim 2$. This result means that for the trapped mode the scattering of the entire sheet of particles can be estimated by computing only the trajectory for the most stably trapped ($\phi_0 = 0$) particle. The slight deviation of some portion of the $\alpha_{eq} = 10^\circ$ curve from the others is due to the fact that the interaction is not entirely in the trapped mode for wave intensities corresponding to that portion (see Fig. 4.4a).
FIGURE 4.10  TOTAL SCATTERING AS A FUNCTION OF WAVE INTENSITY FOR $\alpha_{eq} = 70^\circ$. Other parameters have the values given in Table 1.
FIGURE 4.11 \[|\Delta\alpha_{eq}| \text{ FOR THE MOST STABLY TRAPPED PARTICLE VERSUS RMS SCATTERING.}\]

In the rest of this section, we give results for the scattering of the \( \phi_0 = 0^\circ \) particle, denoting its total pitch angle change by \( \Delta\alpha_{eq} \).

In view of the result given in Fig. 4.11 the rms scattering should be \(-0.5 \times \Delta\alpha_{eq}\).

Figure 4.12 shows the phase \( \phi \) and \( \Delta\alpha_{eq} \) in a typical trajectory for the most stably trapped particle. For this case \( B_w = 50 \text{ mV} \) and \( \Delta\alpha_{eq} = 70^\circ \). The particle is strongly trapped in the potential well of the wave and the phase \( \phi \) goes through many oscillations. As the par-
ticle moves away from the equator the stable points around which
oscillates slowly drifts upward (due to increasing $\omega_H$) until it reaches
$\pi/2$ at $t = 320$ msec at which time the particle becomes detrapped.
During the time the particle is trapped, its pitch angle continuously
decreases to a total net value of $\Delta \omega_{eq} = 40^\circ$ at $t = 320$ msec.
The motion of the trapped particle can be understood by considering
the 'pendulum' equation (2.45) rewritten as

$$\ddot{\phi} + k \left( \frac{eB}{m} \right) v_\perp \sin \phi = \left[ \frac{3}{2} v_\parallel + \frac{k v_\perp^2}{2 \omega_H} \right] \frac{\partial \omega_H}{\partial z}$$

(4.13)

The forcing function of this equation is

$$G(z) = H(t) = \left[ \frac{3}{2} v_\parallel + \frac{k v_\perp^2}{2 \omega_H} \right] \frac{\partial \omega_H}{\partial z}$$

(4.14)

For particles moving away from the equator $\frac{\partial \omega_H}{\partial z}$ continually increases
so that $H(t)$ is an increasing function of time. In a dipole field,
for locations close to the equator the variation of $\omega_H$ is approximated
by (4.7). Using that we obtain

$$H(t) \approx \left[ \frac{3}{2} v_\parallel + \frac{k v_\perp^2}{2 \omega_H} \right] \left( \frac{\partial \omega_H}{\partial z} \right)_{v_\parallel}$$

(4.15)

The restoring force of the 'pendulum' is proportional to $k \left( \frac{eB}{m} \right) v_\perp$. If
the forcing function $H(t) = 0$ (i.e. for a homogeneous field) the phase
$\phi$ oscillates around $\phi = 0$ with a period of $t_T = 2\pi/|\omega|$ where
$\omega^2 = k \left( \frac{eB}{m} \right) v_\perp$. Since $\frac{dv}{dt}$ and $\frac{dx}{dt}$ are both proportional to $\sin \phi$
(see Eq. (2.46a,b) the electron energy and pitch angle would also oscil-
late around $\phi = 0$ with the same period $t_T$. 

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FIGURE 4.12  TRAJECTORY OF A STRONGLY TRAPPED PARTICLE. Note that the pitch angle continuously decreases with $\phi = \pi/2$, after which time $\Delta \alpha_{eq} = 40^\circ$ is constant. Also shown are the corresponding positions of a pendulum with an applied torque $\Delta \alpha_{eq}$ which increases with time. The two problems are fully analogous with the wave and inhomogeneity corresponding to gravity and the applied torque respectively.
In the inhomogeneous case, however, \( \phi = 0 \) is no longer an equilibrium value for \( \Phi \). The equilibrium value \( \Phi_0 \), for which \( \ddot{\Phi} = 0 \), is given by

\[
\Phi_0 = \frac{1}{\omega^2} \left[ \frac{3}{2} V_\perp + \frac{k V_\parallel^2}{2 \omega_H} \right] \frac{\partial \omega_H}{\partial z} = \frac{H(t)}{\omega^2}
\]  

(4.16)

If the right hand side of (4.16) varies slowly compared to \( t_T \), the phase \( \Phi \) oscillates around this new equilibrium value with a period \( t'_T = 2\pi/(\omega \sqrt{\cos \Phi}) \).

For particles moving away from the equator \( \Phi \) slowly increases with time. Initially \( \Phi = 0 \) and therefore \( t'_T = t_T \). Note that the particle is detrapped when \( \phi = \pi/2 \) which must occur before \( \Phi = \pi/2 \) because of the excursions of \( \phi \) around \( \Phi \). Since \( \phi \) stays close to \( \Phi \) for the trapped particles and \( 0 \leq \Phi < \pi/2 \) all trapped particles decrease in energy (see Eq. (2.46a,b)) for this case. For particles moving toward the equator \( \frac{\partial \omega_H}{\partial z} < 0 \) and \( -\pi/2 < \Phi < 0 \), so that the trapped particles increase in energy and pitch angle.

Note that since \( V_\perp \) decreases (see Eq. (2.36b)) during the interaction the restoring force weakens. The oscillating frequency \( \omega \) decreases in time, so that the period, \( t'_T \), varies not only because of varying \( \Phi \) but also because of variation of \( \omega \).

Note that the problem of a strongly trapped particle is completely analogous to the case of a heavy pendulum with an applied torque \( \tau(t) \) such that \( \tau(t) \) is an increasing function. In that case the restoring force is gravity instead of the wave forces and the forcing function is \( \tau(t) \) instead of \( H(t) \). The analogy is depicted in Fig. 4.12 where we show the pendulum position at various places along the particle trajectory.
Accurate analytic treatment of the particle motion taking into account all of these effects is only possible when $\hat{A}(t) t^1_T << H(t)$. For that case one can use an adiabatic approximation by averaging over one oscillation. The computer simulation of the basic equations of motion (2.36) accounts for all of the effects in the general case, even for cases when $\phi$ does not go through one oscillation. An example of this is the particle trajectories given in Fig. 4.2. Those particles are not trapped, but their motion is still described by (4.13) and the analogy with the pendulum is still valid. Note from Fig. 4.12 that all aspects discussed above, including the variation of $t^1_T$ due to changing $v_\perp$, are clearly visible from the computed trajectory.

With these insights into the physics of the trapped particle motion we now survey the pitch angle and energy scattering for different parameters using the $\phi_0 = 0$ particle as our standard for the complete sheet.

Figure 4.13 shows $\Delta \alpha_{eq}$ versus $\alpha_{eq}$ for different wave intensities. For any given wave intensity $\Delta \alpha_{eq}$ peaks around $\alpha_{eq} = 65^\circ$. The scattering for the $\alpha_{eq} = 80^\circ$ particle is less than that for $\alpha_{eq} = 70^\circ$ although at first thought one might have expected a larger $\Delta \alpha_{eq}$. This behavior can be understood with the aid of Eqs. (4.13) - (4.16), in connection with Eq. (2.46b). From Eq. (2.46b) we see the wave induced pitch angle changes are proportional to $\sin \phi$. Assuming many oscillations in the interaction region $\phi = \bar{\phi}$, where $\bar{\phi}$ is given by (4.16). Examining the two terms of $H(t)$ we find that for the parameters of Table 1,
FIGURE 4.13 $|\Delta \alpha_{eq}|$ FOR THE $\phi_0 = 0$ PARTICLE VERSUS $\alpha_{eq}$ FOR THREE DIFFERENT WAVE INTENSITIES. The figure in the upper left corner indicates that the interaction starts at resonance at equator and that the particle is trapped throughout the interaction region with $v_\parallel = v_R$. Pitch angles from the loss cone ($\sim 5^\circ$) to $85^\circ$ are considered. For $\alpha_{eq} > 85^\circ$ the particle energies are in the relativistic range.

\[
\frac{3}{2} v_\parallel > \frac{k v^2_\perp}{2 \omega_H} \quad \text{for } \alpha < 65^\circ
\]

and

\[
\frac{3}{2} v_\parallel < \frac{k v^2_\perp}{2 \omega_H} \quad \text{for } \alpha > 65^\circ
\]
Therefore for \( \alpha < 65^\circ \) we have

\[
H(t) = \frac{3}{2} V_\| \frac{\partial \omega_H}{\partial z}
\]

and

\[
\frac{\partial}{\partial \phi} = \frac{3}{2} V_\| \frac{\partial \omega_H}{\partial z} \cdot \frac{3}{2} V_\| \frac{\partial \omega_H}{\partial z} \cdot \frac{eB}{m} \frac{1}{k(\frac{v}{v_\|})} \tag{4.19}
\]

so that for \( \alpha < 65^\circ \) \( H(t) \) is approximately independent of pitch angle and \( \phi \) decreases with increasing pitch angle. Therefore the distance from the equator to the point where \( \phi = \pi/2 \) (detrapping point) is increased, therefore increasing the interaction length, \( L_I \). The interaction length \( L_I \) is defined as the distance from the equator to the point where \( \phi = \pi \). The wave induced pitch angle scattering is given from (2.46b) as

\[
d\alpha = \left[ \frac{-eB}{m} \sin \phi - \frac{v}{v_\|} \frac{E_w}{v} \cos \alpha \sin \phi \right] dt \tag{4.20}
\]

and the total pitch angle change can be obtained by integrating (4.20),

\[
\Delta \alpha = - \int_0^{T_I} \left[ \frac{-eB}{m} \sin \phi + \frac{v}{m} \frac{E_w}{v} \cos \alpha \sin \phi \right] dt \tag{4.21}
\]

where \( T_I \) is the interaction time given approximately by \( T_I = \int_0^{L_I} \frac{dz}{V_\|} \).

From (4.21) it is obvious that increased interaction length (or time) should result in larger pitch angle changes. Therefore for \( \alpha < 65^\circ \), \( \Delta \alpha_{eq} \) increases with pitch angle. For \( \alpha > 65^\circ \) we have

\[
H(t) = \frac{k v}{2 \omega_H} \frac{\partial \omega_H}{\partial z} \tag{4.22}
\]

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and
\[ \bar{\phi} = \frac{k\nu^2 \omega_H}{2 \omega_H \frac{\partial}{\partial z}} \left( \frac{\partial H}{\partial z} \frac{eB}{m} \right) \cdot (v_L) \] (4.23)

It is evident from (4.23) that for \( \alpha > 65^\circ \), \( \bar{\phi} \) at any point during the interaction is higher for higher pitch angles. Therefore \( \bar{\phi} \) reaches \( \pi/2 \) quicker and the interaction length is shortened, resulting in smaller \( \Delta\alpha_{eq} \).

Note that the above discussion is qualitative. The pitch angle and \( v_L \) change during the interaction so that the variation of \( \bar{\phi} \) could go from that given by (4.23) to that given by (4.19), if the local pitch angle decreases below 65\(^\circ\). Also in (4.20) we have ignored the adiabatic variation in (2.46b). This is an additional complicating factor especially for high pitch angles. Furthermore the approximation (4.7) begins to break down when the interaction length becomes large. In that case \( H(t) \) does not vary linearly in time. Therefore decreased (increased) \( \bar{\phi} \) does not necessarily mean smaller (larger) scattering. One other factor that further complicates the problem can be seen from Eq. (2.46b). The pitch angle change due to \( E_w \) is proportional to \( \cos \alpha \). Since \( v_H \) is fixed by (2.30), resonant particles at higher pitch angles have larger \( v \). Since \( \cos \alpha \) is smaller for higher pitch angles, this effect also contributes to the decreased \( \Delta\alpha_{eq} \) at higher pitch angles. The results given in Fig. 4.13 and the following figures are taken directly from the computer simulation which accounts for all factors. The approximate analytical discussion given above is very useful, however, especially for the qualitative understanding of the results.
Figure 4.14 illustrates the $B_w$ dependence of $\Delta \alpha_{eq}$ that is partially implicit in Fig. 4.13. For each $\alpha_{eq}$, the pitch angle change increases with wave intensity, until $\Delta \alpha_{eq} = \alpha_{eq}$. Shown in a dashed line is the extension of the variation for small wave intensities. For these low wave amplitudes the scattering is in the linear mode. Although the amplitude below which this occurs is dependent on $\alpha_{eq}$, the differences are not noticeable on the scale of Fig. 4.14. Note from this figure that for $B_w \gg 5 m_\gamma$, $\Delta \alpha_{eq}$ is again approximately proportional to $B_w$, although with much steeper slope than that for the linear mode. Also $\Delta \alpha_{eq}$ saturates when $\Delta \alpha_{eq} = \alpha_{eq}$. The distortion in the $\alpha_{eq} = 10^\circ$ one around $B_w = 10 m_\gamma$ is due to the fact that this wave intensity is not sufficient to achieve a trapped mode (see Fig. 4.4a). The slow decrease of $\Delta \alpha_{eq}$ for $B_w > 40 m_\gamma$ for the $\alpha_{eq} = 10^\circ$ particle is a real effect. This effect is due to the "loss cone reflection effect" explained in Appendix B. The sharp minimum of the $\phi_0 = 0$ curves of Fig. 4.3 were also attributed to this effect (see Section 2.A).

Figure 4.15 shows the interaction length $L_I$ as a function of wave intensity for different $\alpha_{eq}$. For $\alpha_{eq} \leq 70^\circ$ the interaction length increases with wave intensity in the same manner as $\Delta \alpha_{eq}$ shown in Fig. 4.14. For $B_w > 30 m_\gamma$, $L_I$ for $\alpha_{eq} = 80^\circ$ is less than that for $\alpha_{eq} = 70^\circ$, as expected from the discussion connected with (4.23). However for $B_w > 30 m_\gamma$ the interaction length for $\alpha_{eq} = 80^\circ$ is larger that that for $\alpha_{eq} = 70^\circ$ although $\Delta \alpha_{eq}$ is smaller as seen from Fig. 4.13. This could result from the fact that for $B_w > 30 m_\gamma$ the pitch angle changes (or changes in $v_\perp$) during the interaction could cause $\phi$ to transform from a variation given by (4.23) to that given by (4.19).
Further evidence for this is the $L_I$ variation for $\alpha_{eq_0} = 85^\circ$. The crossover occurs at $B_w \approx 50 \text{ my}$ for this case as would be expected since for any given $B_w$, $\Delta \alpha_{eq}$ for $\alpha_{eq} = 85^\circ$ is less than that for $\alpha_{eq} = 80^\circ$ (see Fig. 4.13). Note that the interaction length for $B_w = 0$ is ~600 km for $\alpha_{eq_0} = 10^\circ$ and 250 km for $\alpha_{eq_0} = 85^\circ$ in accordance with a pitch angle dependence proportional to $[3 + 0.63 \tan^2 \alpha]^{-1/3}$ as can be seen from (4.10). Note that $L_I \neq z_2$, however $L_I$ can be approximately calculated from Eq. (4.8) by setting $\Delta \phi = \pi$. The slight increases in $L_I$ for $\alpha_{eq_0} = 10^\circ$ at $B_w \approx 50 \text{ my}$ and $\alpha_{eq_0} = 30^\circ$ at $B_w \approx 100 \text{ my}$ are due again to the "loss cone reflection effect" that is discussed in
FIGURE 4.15  INTERACTION LENGTH, $L_I$, VERSUS $B_w$ PLOTTED FOR DIFFERENT $\alpha_{eq}$. 

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Appendix B.

The most important aspect of the result shown in Fig. 4.15 is the fact that for all $\alpha_{eq}$, $L_1$ stays constant beyond same wave intensity. This means that for wave intensities beyond that for which $\Delta \alpha_{eq} \approx \alpha_{eq}$, it is not possible to reach $\phi = \pi$ point any quicker. It is also not possible to have $\Delta \alpha_{eq}$ equal to $\alpha_{eq}$ in a shorter distance. This is at first surprising since it seems from (2.46b) that higher $B_w$ should give larger $d\alpha$. However, with increased wave amplitude $\tilde{\omega}$ is reduced, therefore reducing $\phi$ through (4.16). Since $d\alpha$ in (2.46b) is proportional to $B_w \sin \phi$, the differential changes, initially, are about the same, resulting with the same interaction length and total scattering.

Figure 4.16 shows the percentage energy change ($\Delta K_E$) versus $\alpha_{eq}$ for different wave intensities. The quantity $\Delta K_E$ is defined as

$$\Delta K_E = \left| \frac{K^O_E - K^F_E}{K^O_E} \right| \times 100$$

where $K^O_E$ and $K^F_E$ are the initial (before interaction) and final (after interaction) kinetic energies of the particle. We see that $\Delta K_E$ varies much the same way as $\Delta \alpha_{eq}$ shown in Fig. 4.13. The peak around $\alpha = 65^\circ$ could be qualitatively explained by using arguments similar to those used for that figure.

Note that although the percentage energy change decreases at higher pitch angles, the absolute value of energy change should continue to increase since energy exchange occurs only through $v_\perp$ which is higher at higher pitch angles. This is illustrated in Fig. 4.17 where we plot
FIGURE 4.16  PERCENTAGE ENERGY CHANGE, $\Delta K_E$, FOR THE $\phi_0 = 0$ PARTICLE VERSUS $\alpha_{eq}$ PLOTTED FOR DIFFERENT $B_w$. 
the total absolute energy change for the particle in electron volts (eV) versus $\alpha_{eq}$ for different values of $B_w$. We observe that total absolute energy change increases with both $B_w$ and $\alpha_{eq}$.

Finally in Fig. 4.18 we show the wave amplitude dependence of $\Delta K_E$ which is implicit in Fig. 4.16. We see that this variation is very similar to that of $\Delta \alpha_{eq}$ in Fig. 4.14.
Figure 4.18 Percentage energy change $\Delta K_e$ versus $B_w$ plotted for different $\alpha_{eq}$.

Note from Fig. 4.14, that, for instance, for $\alpha_{eq} = 30^\circ$ and $B_w = 100$ mG, $\Delta \alpha_{eq} = \alpha_{eq}$, meaning that the particle has lost all of its perpendicular energy. The initial perpendicular energy of this particle was 25% of its total energy since $v_\perp^2 = (v \sin 30^\circ)^2 = 0.25v^2$. However, Fig. 4.18 shows that the particle has lost only 8% of its total initial energy. That the particle's total initial perpendicular energy is not extracted can be understood as follows: During the process of the strongly trapped interaction $v_\parallel = v_R$, where $v_R$ is the local resonance
velocity. Since $v_R$ increases as the particle moves away from the equator, particles parallel energy must increase during the interaction, resulting in the extraction of less than the initially available perpendicular energy of the particle. One can loosely define an energy extraction efficiency as the ratio of the extracted and initially available perpendicular energies. For $\alpha_{eq} = 30^\circ$ that efficiency is $\sim 32\%$. For $\alpha_{eq} = 70^\circ$ available energy is $88\%$, while only $32\%$ is extracted for $B_w = 150$ mT for which $\Delta\alpha_{eq} = 60^\circ$. So the energy extraction efficiency for that case is $\sim 36\%$. For $\alpha_{eq} = 10^\circ$ the extraction efficiency is $\sim 33\%$. Therefore we can conclude that for the values used in our computations and approximately independent of pitch angle the cyclotron resonance interaction is $\sim 30\%$ efficient in extracting the particle's perpendicular energy. This is true regardless of the wave intensity as long as $B_w$ is large enough so that $\Delta\alpha_{eq} = \alpha_{eqo}$. Note that in terms of pitch angle the interaction is 100\% efficient, since for any $\alpha_{eqo}$ a $B_w$ could be found for which $\Delta\alpha_{eq} = \alpha_{eqo}$. Again wave intensities beyond that one cannot do any better.

\[ \rho \] as an empirical criteria

We have indicated in section 3.8 that the correlation between the pitch angle dependence of $\rho$ and the variation of $\Delta\alpha_{eq}$ vs. $\phi_0$ curves given in Fig. 4.7 was interesting and could mean that $\rho$ can be used as an indication of absolute scattering for a given case as well as a criterion for linear theory.

Figure 4.19 shows rms scattering $\sqrt{\Delta\alpha_{eq}^2}$ versus $\rho$ for $\alpha_{eqo} = 30^\circ, 50^\circ$ and $70^\circ$. In this case $\rho$ is varied by varying $B_w$. The dashed line is drawn to indicate the average trend. Note that the var-

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FIGURE 4.19 \( \text{rms SCATTERING } \sqrt{\langle (\Delta \alpha_{eq})^2 \rangle} \) VERSUS \( \rho \). The dashed line shows the average trend.

\( \alpha_{eq} \) variations for different \( \alpha_{eq} \) do not deviate more than 20% from the dashed line. Hence in any given problem where \( B_w, k, \frac{\partial H}{\partial z} \) and is specified, one can compute \( \rho \) using (4.4) and find the rms scattering from Fig. 4.19 within an error of 20%. It could be possible to increase the accuracy further by multiplying by some appropriate functions of pitch angle. This seems possible since comparing Figs. 4.14 and 4.19 we see that the pitch angle dependence of \( \rho \) brings the curves for different \( \alpha_{eq} \) together in the right direction but rather too much probably because it is computed in terms of initial parameters and the pitch angle significantly changes during the interaction.
V. PRECIPITATED FLUX FROM A FULL DISTRIBUTION OF PARTICLES

In this chapter we present an application of our simulation for the calculation of the one-pass precipitated electron flux for a particular case.

A. SIMULATION OF THE FULL DISTRIBUTION

The full distribution of particles is simulated by a large number of test particles. For energetic particles adiabatically trapped in the earth's magnetic field, the particles population in every flux tube can be represented by an equatorial distribution function \( f_{eq}(\psi_{eq}, \alpha_{eq}) \).

From this point on we drop the subscript 'eq' for the purpose of simplifying the text. Unless otherwise mentioned all quantities \( \psi \) and \( \alpha \) in this chapter represent equatorial values. We have chosen to work with the distribution function in the \( \psi - \alpha \) space as opposed to \( v^\parallel - \alpha \) or \( v^\parallel - v^\perp \) spaces. This formulation is convenient for our simulation since it directly shows the pitch angle scattering along the axis and makes it possible to uniquely identify each \( \psi^\parallel \) mesh point with a resonance location through (2.30). The velocity space volume element in terms of \( \psi^\parallel \) and \( \alpha \) is \( \psi^\parallel \frac{\sin \alpha}{\cos^3 \alpha} \, d\psi \, dv^\parallel \, d\phi \) (see Appendix C).

For a given field line and a given wave, only a limited portion of the total particle population represented by this distribution will resonate with and hence be scattered significantly by the wave. Therefore, in our simulation, we need only consider that limited portion of the distribution function. This is the shaded area shown in Fig. 5.1a. The \( \alpha_{min} \) is determined by the loss cone. For given plasma and wave parameters \( \alpha_{max} \) is determined by the nature of the problem. For
FIGURE 5.1 SIMULATION OF THE DISTRIBUTION FUNCTION. (a) The general distribution. The shaded area is the portion that will be significantly affected by the wave. (b) Unperturbed distribution. In this illustration a uniform distribution with \( f(v_{\|}, \alpha) = 12 \) above the loss cone and \( f(v_{\|}, \alpha) = 0 \) in the loss cone is shown. (c) Perturbed distribution.
example, for the computation of the one-pass precipitated flux, an
\( \alpha_{\text{max}} < \pi/2 \) can always be found such that for the given parameters, par-
ticles with \( \alpha > \alpha_{\text{max}} \) cannot be scattered into the loss cone at the
first encounter with the wave. On the other hand, if one wishes to con-
sider the steady state scattering problem, it is necessary to take
\( \alpha_{\text{max}} \approx 90^\circ \) since in this case multiple scatterings can bring high pitch
angle particles down to the loss cone. The \( v_{\parallel \text{min}} \) is determined by
the resonance velocity at the equator. Particles with parallel vel-
cities lower than \( v_{\parallel \text{min}} \) do not resonate with the wave at any point
along the field line. The upper limit \( v_{\parallel \text{max}} \) is determined by the
fact that particles with higher \( v_{\parallel} \) resonate at points so far down the
field line that the scattering produced is less than one half of the
pitch angle mesh size used in the calculations. That is for given para-
eters we can find a \( v_{\parallel \text{max}} \) such that the scattering for particles with
\( v_{\parallel} > v_{\parallel \text{max}} \) will be negligible and need not be considered. Once the
shaded area is determined, this region in the \( v_{\parallel} - \alpha \) space is divided
into a number of mesh points. Each mesh point is identified with a pair
\( v_{\parallel} \) and \( \alpha \). The value of the distribution at each mesh point is
\( f(v_{\parallel}, \alpha) \). Figure 5.1b illustrates such a representation of the distribu-
tion function. For purposes of illustration we have chosen a uniform
distribution with a sharp cutoff at the loss cone. The mesh sizes in
\( v_{\parallel} \) and \( \alpha \) are \( \Delta v_{\parallel} \) and \( \Delta \alpha \) respectively. The number density of
particles with parallel velocities \( v_{\parallel} + \Delta v_{\parallel} \) and pitch angles
\( \alpha + \Delta \alpha/2 \) is given by

\[
\Delta N = 2\pi f(v_{\parallel}, \alpha) v_{\parallel}^2 \frac{\sin \alpha}{\cos^3 \alpha} \Delta \alpha \Delta v_{\parallel}
\]  
(5.1)

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where the factor $2\pi$ is due to integration over the cyclotron phase $\phi$. We assume a uniform distribution in $\phi$.

In our formulation each such population $(v_{||} \pm \frac{\Delta v_{||}}{2}, \alpha \pm \frac{\Delta \alpha}{2})$ of particles is represented by test particles with parallel velocity $v_{||}$ and pitch angle $\alpha$. When the mesh sizes $\Delta v_{||}$ and $\Delta \alpha$ are chosen adequately small, the motion of these test particles accurately represent the motion of all particles with $v_{||} \pm \frac{\Delta v_{||}}{2}$ and $\alpha \pm \frac{\Delta \alpha}{2}$.

Since the gyroresonant interaction is highly dependent on initial phase we use 12 test particles uniformly distributed in $\phi$ for each mesh point. In that case each test particle with initial equatorial values $v_{||0}, \alpha_0$ and a certain phase $\phi_0$ represents the population of particles with

$$v_{||} = v_{||0} \pm \frac{\Delta v_{||}}{2}$$

$$\alpha = \alpha_0 \pm \frac{\Delta \alpha}{2}$$

$$\phi = \phi_0 \pm \frac{\pi}{12}$$  \hspace{1cm} (5.2)

The actual number density of particles described by (5.2) is

$$\frac{\Delta N}{12} = f(v_{||0}, \alpha_0) v_{||0}^2 \frac{\sin \alpha_0}{\cos^3 \alpha_0} \frac{\Delta \alpha \Delta v_{||} (\pi/6)}{6}$$  \hspace{1cm} (5.3)

The next step is to simulate the interaction for the test particle. The test particle is allowed to go through the complete interaction as described in Chapter 3. In other words the equations of motion for the test particle are integrated from the point $N'$ in the northern hemisphere where $\epsilon = \frac{v_{||0} - v_R}{v_R} = -0.03$ till the point $S'$ in the southern hemisphere where $\epsilon = +0.05$ (see Fig. 3.2b). At the end of the inter-
action the test particle has acquired a new equatorial velocity and pitch angle, namely \( v_{\parallel F} \) and \( \alpha_F \) (F for final) and must now be identified with the mesh point \((v_{\parallel F}, \alpha_F)\). This in effect means that the number of particles described by (5.3) have all acquired parallel velocity and pitch angle values in the ranges \( v_{\parallel F} \pm \frac{\Delta v_{\parallel}}{2} \) and \( \alpha_F \pm \frac{\Delta \alpha}{2} \) respectively. In order to conserve the total number density of particles in the system the following changes must be made in the values of the distribution function at \((v_{\parallel 0}, \alpha_0)\) and \((v_{\parallel F}, \alpha_F)\).

\[
f_{\text{new}}(v_{\parallel 0}, \alpha_0) = f_{\text{old}}(v_{\parallel 0}, \alpha_0) - \frac{1}{12} f_{\text{old}}(v_{\parallel 0}, \alpha_0)
\]

\[
f_{\text{new}}(v_{\parallel F}, \alpha_F) = f_{\text{old}}(v_{\parallel F}, \alpha_F) + \frac{1}{12} f_{\text{old}}(v_{\parallel 0}, \alpha_0) \left( \frac{v_{\parallel F}^2}{\cos^3 \alpha_F} \right) \left( \frac{\sin \alpha_F}{\sin \alpha_0} \right)
\]

Using 12 such test particles for each mesh point, and repeating the procedure for every mesh point, we obtain the perturbed distribution.

Figure 5.1c shows an illustrative sketch of the perturbed distribution. After one pass of the wave the empty loss cone of the initial distribution is partly filled. The total number of precipitated particles and the precipitated flux can be obtained by properly integrating the perturbed distribution over \( v_{\parallel} \) and \( \alpha \). This is done in detail for the sample computation of the next section.
B. COMPUTATION OF FLUX FOR A PARTICULAR CASE

As an application of the technique described above, we compute the precipitated particle flux for a particular case. The medium and wave parameters given in Table 1 are used for different wave intensities. For these computations a pitch angle mesh size of 0.5 degrees is used, with 60 mesh points in pitch angle covering from \( \alpha = 0^\circ \) to \( 30^\circ \) in 0.5\(^\circ\) steps. Pitch angles greater than 30\(^\circ\) were not considered because even for the largest wave intensity used in the calculations \( \beta = 50 \) m\( \gamma \) no particles with \( \alpha_0 > 30^\circ \) are scattered into the loss cone during one pass through the wave. A variable mesh size in \( v_\| \) is used with mesh size increasing with \( v_\| \). About 85 bins in \( v_\| \) covering the range from \( v_n = 1.87 \times 10^4 \) km/sec (parallel energy \( \sim 1 \) keV) to \( v_\| = 6.0 \times 10^4 \) km/sec (parallel energy \( \sim 10 \) keV) were used with mesh sizes ranging from 0.1\% to 2.8\%. The resonance points are at the equator for the 1 keV particles and at \( \pm 20^\circ \) for particles at 10 keV parallel energy (assuming \( \alpha_{eq} = 15^\circ \)). Altogether about 40,000-50,000 test particles are used for each computation.

The initial unperturbed distribution is taken to be of the form

\[
f(v,\alpha) = \frac{A}{v^4} g(\alpha) \tag{5.4}
\]

where \( A \) is a constant and \( g(\alpha) \) is some function of pitch angle. The energy variation of \( f(v,\alpha) \) is reasonable compared to particle data [Schield and Frank, 1970]. In our computations we consider two different distribution functions
(a) an isotropic distribution for which

\[
g(\alpha) = g_1(\alpha) = \begin{cases} 
1 & \alpha > 1c \\
0 & \alpha < 0 
\end{cases}
\]  
\tag{5.5}

and

(b) an anisotropic distribution with

\[
g(\alpha) = g_2(\alpha) = 0.2 \sin^2 \alpha + 0.8 \sin^{12} \alpha & \quad \alpha > 1c \\
= 0 & \quad \alpha < 1c
\]  
\tag{5.6}

The two distributions have the same value at \( \alpha = 90^\circ \) and they are sketched in Fig. 5.2. Both isotropic and anisotropic distributions have been measured in the magnetosphere [Lyons and Williams, 1975]. The particular anisotropic distribution of Fig. 5.2b was measured by Anderson [1976] and is considered to be highly anisotropic.

Note that both distributions of Fig. 5.2 are essentially flat for \( 0 < \alpha < 30^\circ \). In our computations of the precipitated flux due to one pass through the wave we only consider particles with \( \alpha < 30^\circ \). Therefore for the purposes of the computer simulation both distributions can be treated as isotropic, but with different flux levels above the loss cone.

(a) isotropic case

In this case the initial unperturbed distribution is

\[
f(v_\parallel, \alpha) = \frac{A \cos^4 \alpha}{v_\parallel^4} & \quad \text{for } \alpha > 1c \\
= 0 & \quad \text{for } \alpha < 1c
\]  
\tag{5.7}

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FIGURE 5.2  PITCH ANGLE DEPENDENCE OF THE TWO DISTRIBUTION FUNCTIONS USED IN THE CALCULATIONS. (a) an isotropic distribution, (b) an anisotropic distribution.
With this as the input distribution, the test particle simulation described in the last section is carried out and the perturbed distribution is obtained. The output of this computation is the value of the perturbed distribution at the 30x85 = 2550 mesh points, covering from $\alpha = 0^\circ$-$30^\circ$ and $v_\parallel = v_k$ to $v_\parallel = \sqrt{10} v_k$ where $v_k = 1.9 \times 10^7$ m/sec is the velocity corresponding to 1 keV energy, such that $0.5 m v_k^2 \approx 1$ keV.

In the following, we concentrate on the 1-2 keV total energy band. First, in order to show the wave induced pitch angle perturbations of the distribution function we integrate the distribution over $v_\parallel$, to obtain:

$$f(\alpha) = 2\pi \int_{v_\parallel = v_k \cos \alpha}^{v_\parallel = v_k^2 \cos \alpha} f(v_\parallel, \alpha) v_\parallel^2 dv_\parallel$$

(5.8)

where we again assume uniform distribution over $\phi$.

Figure 5.3 shows the normalized distribution $f(\alpha)$ vs. $\alpha$, for different wave intensities. The dashed lines show the unperturbed distribution which is isotropic above the loss cone as is clear from (5.7). The solid lines show the one-pass perturbed distribution. Note again that only the 1-2 keV total energy band is considered. The integral given in (5.8) is easily carried out with the computer, using the mesh point values $f(v_\parallel, \alpha)$ and $\Delta v_\parallel = dv_\parallel$, and approximating the continuous integral by a weighted sum of finite number of values of $f(v_\parallel, \alpha)$.

For $B_w = 1$ mG the perturbations are small and only a small percentage of the particle population from the range just above the loss cone have been precipitated. For $B_w = 10$ mG we see that the loss
FIGURE 5.3 NORMALIZED PARTICLE DISTRIBUTION $f(\alpha)$. The dashed lines represent the unperturbed distribution which is isotropic in pitch angle. The solid lines represent the one-pass perturbed distribution. Note that the number density in any given range $d\alpha$ is equal to $f(\alpha)\sin\alpha d\alpha$. 
cone is partly filled with particles scattered down from higher pitch angles. As can be seen from the figure most of the particles scattered into the loss cone originally had pitch angles in the 2-3° range just above the loss cone. For \( B_w = 20 \text{ mG} \) and \( 50 \text{ mG} \) we observe that the number of precipitated electrons is higher than for \( B_w = 10 \text{ mG} \) and there are more particles deeper into the loss cone. Also by observing the distribution at pitch angles above the loss cone we see that contributions to the loss cone population come from a wider range of pitch angles for higher wave intensities.

The total number density of precipitated electrons in the 1-2 keV energy range is given by:

\[
N_T = 2\pi \int_{v_{\parallel} = v_k \cos \alpha}^{1c} \sqrt{2} v_k \cos \alpha \int_{0}^{\alpha_{1c}} f(v_{\parallel}, \alpha) v_{\parallel}^2 \frac{\sin \alpha}{\cos^3 \alpha} \, dv_{\parallel} \, da \left[ L^3(1+3\sin^2 \lambda_i)^{1/2} \right] \quad (5.9)
\]

where \( \alpha_{1c} \) is the loss cone angle (5.5° at equator for \( L = 4 \)) and the last factor \( L^3(1+3\sin^2 \lambda_i)^{1/2} \) is due to the convergence of the field lines which gives a reduced flux tube cross section at the ionosphere as compared to that at the equator. In this case \( \lambda_i \) would be the latitude at which the \( L = 4 \) field line crosses ionospheric heights, i.e., \( \lambda_i = 60° \). Note that all evaluations in the integrand are done using equatorial values whereas \( N_T \) is the precipitated number density at the ionosphere. The precipitated energy deposition rate in the same range is:
\[ Q = 2\pi \int_0^1 \int_0^{\sqrt{2}v_k \cos \alpha} f(v_\parallel, \alpha) v_\parallel^2 \frac{\sin \alpha}{\cos^3 \alpha} \left( \frac{1}{2m} - \frac{v_\parallel^2}{c^2} \right) dv_\parallel d\alpha \frac{dL^3(1 + 3 \sin^2 \lambda_1)^{1/2}}{c^2} \]

(5.10)

Both integrations (5.9) and (5.10) reduce to finite weighted sums in our computer formulation. With the perturbed distribution \( f(v_\parallel, \alpha) \) obtained at the mesh points, \( N_T \) and \( Q \) are evaluated without difficulty. For \( B_w = 10 \text{ mT} \) we obtain

\[ N_T = 9.5 \times 10^{-11} \text{ A cm}^{-3} \]  

(5.11)

and

\[ Q = 4.2 \times 10^{-10} \text{ A ergs/cm}^2\text{-sec} \]  

(5.12)

where \( A \) is the proportionality constant in (5.4) and (5.7).

The constant \( A \) can be evaluated in terms of measured number density as follows. Assume a total number density in the 1-2 keV energy range of \( N_e \text{ el/cc} \) isotropically distributed in pitch angle. Then,

\[ N_e = 2\pi \int_0^\pi \int_0^{\sqrt{2}v_k} \frac{A}{v^4} v^2 \sin \alpha dv d\alpha \]

(5.13)

From the above we obtain

\[ A = 5.2 \times 10^8 N_e \]

(5.14)

Substituting (5.14) in (5.11) and (5.12) we have

\[ N_T = 0.05 N_e \text{ cm}^{-3} \]

(5.15)
\[ Q = 0.2 N_e \text{ ergs/cm}^2\text{-sec} \quad (5.16) \]

The above results are for \( B_w = 10 \text{ mT} \). Figure 5.4 shows the energy deposition rate as a function of wave intensity. The dashed straight line gives the results predicted on the basis of linear theory. The vertical scale on the left is normalized to \( |N_e| \). The one on the right is normalized to \( |\varphi_1| \), the differential energy spectrum for \( \alpha = 90^\circ \) particles with \(-1 \) keV energy. As seen from Fig. 5.4 the precipitated flux increases with wave intensity for \( B_w < 40 \text{ mT} \) and begins to saturate for \( B_w > 40 \text{ mT} \). The variation should be compared to that of the rms scattering given in Fig. 4.6.

Schield and Frank [1970] have reported measurements of low energy electrons on the OGO-3 satellite. Figure 6 of their paper indicates number densities of \(3 \) el/cc in the energy range \(750 \text{ eV} \leq E \leq 50 \text{ keV} \) inside the plasmapause at \( L = 4 \). Figure 4 of the same paper indicates that number density varies as \( \varphi^{-4} \) with velocity \( (E^{-2} \) with energy). Using this type of energy dependence we obtain \( N_e = 1 \) el/cc in the 1-2 keV range. Using (5.15) and (5.16) we then have a precipitated number density of \( N_T = 0.05 \text{ cm}^3 \) and a flux of \( Q = 0.22 \text{ ergs/cm}^2\text{-sec} \) for \( B_w = 10 \text{ mT} \) wave intensity.

The energy deposition rate can also be expressed in terms of the differential energy spectrum [Schield and Frank, 1970]. The differential energy spectrum \( \phi \) in \text{el/cm}^2\text{-ster-sec-keV} \ is given by

\[ \phi = \frac{f(v,\alpha)\varphi^2 v dv \sin \alpha d\alpha d\Omega}{d\Omega \, dE} \quad (5.17) \]
FIGURE 5.4 THE ONE-PASS PRECIPITATED FLUX IN THE 1-2 keV RANGE AS A FUNCTION OF WAVE INTENSITY. This result is for an isotropic distribution. The vertical scale on the left gives the flux (ergs/cm$^2$-sec) normalized to $|N_e|$ where $N_e$ is the electrons per cm$^3$ in the 1-2 keV range. The one on the right gives flux normalized to $|\phi_1|$, the differential energy spectrum in el/cm$^2$-ster-sec-keV, for ~1 keV electrons at $\alpha = 90^\circ$.

where $d\Omega = \sin\alpha d\alpha d\phi$ and $dE = mv dv$ are the solid angle and energy differentials respectively. In the case of an isotropic distribution $f(v,\alpha) = A/v^4$ and $\phi$ is the same for all pitch angles. Substituting above we obtain

$$A = 2 \phi_1$$

(5.18)
where \( \phi_1 \) is the differential energy spectrum in \( \text{eV/cm}^2 \text{-ster-sec-keV} \) for the \(-1\) keV electrons. Substituting (5.18) in (5.11) and (5.12) we have

\[
N_T = 1.9 \times 10^{-10} \phi_1 \text{ cm}^{-3}
\]

(5.19)

and

\[
Q = 1.0 \times 10^{-9} \phi_1 \text{ ergs/cm}^2 \text{-sec}
\]

(5.20)

for \( B_w = 10 \text{ mG} \). The energy deposition rate as a function of wave amplitude is shown in Fig. 5.4, where the vertical scale on the right is normalized to \( \phi_1 \).

A more recent measurement of low energy electron fluxes on the Explorer 45 \( (S^3) \) satellite was reported by Anderson [1976]. For disturbed pre-midnight conditions and for \(-1\) keV electrons with isotropic distributions the energy spectrum \( \phi_1 \) is of the order of \(-10^8\) \( \text{eV/cm}^2 \text{-ster-sec-keV} \). Substituting in (5.20) we obtain energy deposition rate \( Q \approx 0.1 \text{ ergs/cm}^2 \text{-sec} \).

(b) **anisotropic case**

For this case the initial unperturbed distribution is given as

\[
f(v_\parallel, \alpha) = \frac{A \cos^4 \alpha}{v_\parallel^4} \left[ 0.2 \sin^{0.2} \alpha + 0.8 \sin^{12} \alpha \right]
\]

for \( \alpha > \alpha^{1c} \)

(5.21)

\[
= 0
\]

for \( \alpha < \alpha^{1c} \)

The pitch angle dependence \( g(\alpha) = 0.2 \sin^{0.2} \alpha + 0.8 \sin^{12} \alpha \) is plotted in Fig. 5.2. For \( \alpha < 30^\circ \) this distribution can be approximated by
\[ f(v, \alpha) = (0.15)A \frac{\cos^4 \alpha}{v_{\parallel}^4} \quad \alpha^c < \alpha < 30^\circ \]
\[ = 0 \quad \text{if} \quad \alpha < \alpha^c \quad \text{(5.22)} \]

When the distribution given by (5.23) is used as the initial distribution in our simulation, the results should be the same as those for the distribution given by (5.7) within a factor 0.15. Therefore we have for this anisotropic case and for \( B_w = 10 \text{ m}\gamma \),

\[ N_T = 9.48 \times 10^{-11} \times (0.15)A = 1.42 \times 10^{-11}A \quad \text{cm}^{-3} \quad \text{(5.23)} \]

and

\[ Q = 4.23 \times 10^{-10} \times (0.15)A = 6.35 \times 10^{-11} A \quad \text{ergs/cm}^2\text{-sec} \quad \text{(5.24)} \]

where \( A \) is the proportionality constant in (5.9) or (5.23).

The differential energy spectrum (defined in (5.17)) for the distribution given in (5.21) is:

\[ \phi = \frac{A}{2E} [0.2\sin^{0.2} \alpha + 0.8\sin^{12} \alpha] \quad \text{(5.25)} \]

where \( E \) is the energy. Substituting \( E = 1 \text{ keV} \) and \( \alpha = 90^\circ \), we obtain:

\[ A = 2 \phi_1 \quad \text{(5.26)} \]

where \( \phi_1 \) is the spectrum in el/cm\(^2\)-ster-sec-keV for \(-1 \text{ keV} \) electrons at \( \alpha = 90^\circ \). That this is the same relation as (5.18) is not surprising since the two distributions shown in Fig. 5.2 were chosen to have the same value at \( \alpha = 90^\circ \). Substituting (5.26) in (5.23) and (5.24) we have for \( B_w = 10 \text{ m}\gamma \).
\[ N_T = 2.8 \times 10^{-11} \ \phi_1 \ \text{cm}^{-3} \]  \hspace{1cm} (5.27)

\[ Q = 1.3 \times 10^{-10} \phi_1 \ \text{ergs/cm}^2\text{-sec} \]  \hspace{1cm} (5.28)

We again use the data reported by Anderson to find absolute values for the energy deposition rate. For -1 keV electrons with anisotropic distributions similar to that of Fig. 5.2, the differential energy spectrum for \( \alpha = 90^\circ \) particles \( \phi_1 \approx 10^8 \ \text{cm}^{-2}\text{ster}^{-1}\text{sec}^{-1}\text{keV}^{-1} \). Substituting in (5.28) we obtain \( Q = 1 \times 10^{-2} \ \text{ergs/cm}^2\text{-sec} \).

These values show that the one-pass fluxes in the 1-2 keV range precipitated by a 10 mG CW signal at 5 kHz can be as much as \( 1 \times 10^{-2} \) to \( 0.2 \ \text{ergs/cm}^2\text{-sec} \) depending on the anisotropy of the distribution function. Such fluxes are well within the resolution of most particle detectors. The intensities approach that of a moderate aurora.

C. LEVERAGE

As was shown in section 2.E, large pitch angle changes induced by the wave on particles do not necessarily require a large amount of energy exchange in the cyclotron resonance interaction. The basic reason for this is the fact that the wave perturbations are mainly through the wave magnetic field which changes the direction of momentum (i.e., pitch angle) of the particle without energy transfer.

We have shown in the previous section that energy fluxes of as much as 0.2 ergs/cm\(^2\)-sec can be precipitated by waves of 10 mG intensity. It is instructive to compare the energy density of the precipitated flux and that of the input wave and compute the leverage involved in the
wave induced precipitation process.

For longitudinal whistler-mode propagation the wave Poynting flux is given by

\[ |\overrightarrow{P}| = |E_\perp \times H_\perp| = |E_\perp||H_\perp| = \left| k\frac{B_w}{\nu_0} \right| \frac{B_w}{\nu_0}^2 \]

(5.29)

Using \( n = \frac{kC}{\omega} \), (5.29) can be expressed in terms of the refractive index as

\[ |\overrightarrow{P}| = \frac{C}{n\nu_0} \left| B_w \right|^2 = \frac{2.5 \times 10^{-10}}{n} \left| B_w \right|^2 \text{ watts/m}^2 \]

(5.30)

where \( B_w \) is in milligammas. For the parameters of Table 1 used for our computations, the refractive index \( n = 40 \) at the equator. Using this together with a \( B_w = 10 \text{ mG} \) we obtain from Eq. (5.30) the wave energy flux

\[ |\overrightarrow{P}| = 6 \times 10^{-9} \text{ watts/m}^2 \]

A precipitated flux of 0.2 ergs/cm\(^2\)-sec is equivalent to an energy density of about 0.2x10\(^{-3}\) watts/m\(^2\). This shows that the leverage involved in this interaction is \(-10^5\) or 50 dB. Therefore significant particle fluxes can be precipitated by waves of moderate intensity.

The above calculations have compared the energy densities. We can also consider the total input and output power and compute the leverage on an integrated basis. The Siple VLF transmitter operates with a total radiated power ranging from 100 W to 1 kW. Assuming that the precipitated flux is distributed over an area which is 100 km in radius, the total precipitated power is \(-10^7\) watts, a factor of \(-10^5\) larger than the radiated power. Hence the power leverage of the interaction is \(-50\) dB. The input power of the transmitter is \(-100\) kW. Since an output of
-10^{7} \text{ watts} \text{ is obtained the net power gain is } -20 \text{ dB. The source of this extra power is of course the energy of the trapped energetic particles.}

Note that these numbers are only considering the 1-2 \text{ keV energy band. The same wave will also precipitate particles with higher energies. Therefore the total leverage and the power gain are likely to be larger than those computed above. Furthermore we have implicitly assumed that 100 \text{ kW of input power is necessary for generating a 10 mY signal. This is very likely to be an over estimate. Recent results from a power-step (up and down) experiment using the Siple transmitter has showed that the same magnetospheric signals could be produced with lower power levels [R. A. Helliwell, private communication].}

D. IONOSPHERIC EFFECTS

In this section we investigate the effects in the nighttime ionosphere of the wave induced precipitation fluxes computed in the previous sections.

The precipitating electrons impinge on the lower ionosphere, where they produce numerous secondary electrons and create an impulsive ionization enhancement throughout the volume of the precipitation region. As a result the ionospheric conductivity in the same volume is also enhanced. The incoming electrons with higher energies penetrate to lower altitudes and produce Bremsstrahlung x-rays which are detected by balloon measurements.

The ionospheric density enhancements produced by an incoming flux of electrons at a given energy can be roughly estimated using the results of Banks et al. [1974] and Bailey [1968]. Figure 9 of Banks et al.
[1974] gives the ionization rates per unit incident flux for monoenergetic fluxes in the range 0.42-10 keV. Bailey [1968] gives the ion pair production rates for energies up to 150 keV.

In previous sections we have computed an energy deposition rate of 0.2 ergs/cm$^2$-sec for electrons in the 1-2 keV energy range. This flux can be approximated as a monoenergetic beam of precipitating electrons of energy 1.5 keV.

The flux computation of section 7.8 was done for particles with 1-10 keV parallel energy, but in that section we discussed only the 1-2 keV flux. The precipitated flux for higher energies can also be obtained from the same computer output. For instance, in the 6.5 to 7.5 keV range (average energy 7 keV) the flux is $6 \times 10^{-2}$ ergs/cm$^2$-sec. Although the simulation is carried out only up to 10 keV we can assume linear theory and extrapolate to higher energies. This assumption is justified since we have shown that linear theory applies for resonances sufficiently far from the equator. Particles with parallel energies greater than 10 keV resonate at latitudes beyond 20° and linear theory can be comfortably used. In linear theory the rms scattering is approximately proportional to $\frac{1}{V_\parallel}$ or $(E)^{-1}$ where $E$ is the parallel energy. The precipitated flux is not reduced by $\frac{1}{V_\parallel}$, since for the same number density, particles with higher energy constitute a larger flux. Extrapolating from the computed 7 keV flux to 30 keV using linear theory we obtain $2 \times 10^{-2}$ ergs/cm$^2$-sec, in the 29.5 to 30.5 keV range.

Figure 5.5 shows the ionospheric density enhancements produced by these three computed monoenergetic incoming fluxes with energies 1.5±0.5 keV, 7±0.5 keV and 30±0.5 keV. The density enhancements are
FIGURE 5.5 PRECIPITATION INDUCED DENSITY ENHANCEMENTS PRODUCED BY THREE MONOENERGETIC INCOMING FLUXES AT 1.5, 7 AND 30 keV. The solid lines represent the enhancements due to a precipitation pulse of one second duration. The steady state due to 1.5 keV flux is also shown.

computed assuming a one-second duration of precipitation of these monoenergetic beams. It is assumed that there is negligible recombination in this brief period. We have used the results of Banks et al. [1974] and Bailey [1968] to roughly estimate the density enhancements. We have also given the steady state density enhancement for the 1.5 keV flux assuming a flux of 0.2 ergs/cm²·sec is continuously impinging on the ionosphere. For the steady state computation we have used for altitudes
up to 200 km, the relation

\[ \frac{dN}{dt} = q - \psi N^2 \]

where \( N \) is the electron density, \( q \) is the volume rate of production of electron ion pairs and \( \psi \) is the effective loss coefficient (due to recombination and attachment) for electrons. We have used a value of \( \psi \) of about \( 2 \times 10^{-7} \text{ cm}^3/\text{sec} \) in the 80-200 km altitude range [Bailey, 1968]. For altitudes above 200 km the continuity equation (neglecting diffusion) becomes

\[ \frac{dN}{dt} = q - \beta N \]

and the loss coefficient \( \beta \) is in the range \( 10^{-4} - 10^{-3} \text{ cm}^3/\text{sec}^{-1} \) [Park and Banks, 1974]. These values are used in our calculations of the steady state flux for altitudes above 200 km.

Note that a 5 kHz, 10 mV wave at \( L = 4 \) will precipitate electrons at all energies above 1 keV. We have shown in Fig. 5.5 the perturbations due to only three monoenergetic components of the incoming flux. Using a more accurate computation the total enhancement profile can be obtained from an incoming precipitated flux with given energy spectrum. Elaborate computer programs which give the ionospheric density enhancements due to a given spectrum of precipitated flux do exist and are widely used for studying the penetration of the auroral electrons into the atmosphere [Walt et al., 1969; Banks et al., 1974].

For comparison, an average typical nighttime electron density profile [Hanson, 1961] is also shown in Fig. 5.5. These results show that significant perturbations in the nighttime electron density over the 80-250 km altitude range can be produced by one-second duration
precipitation pulses. The perturbation caused by the 1.5 keV flux could be measured with ionosondes. The enhancements caused by the 30 keV flux occur at low altitudes and could cause detectable modifications in the subionospheric VLF propagation [Helliwell et al., 1973]. These density enhancements also result in significant enhancements in the ionospheric conductivity and could conceivably give rise to micropulsation activity [Bell, 1976].

In addition to density enhancements the computed precipitation fluxes of $-10^{-1}$ ergs/cm$^2$-sec are intense enough to cause enhanced atmospheric photo emission which could be measured by photometers.
VI. CONCLUSIONS AND DISCUSSION

A. SUMMARY

We have analyzed the nonlinear gyroresonance interaction between energetic electrons and coherent VLF waves in the magnetosphere. This mechanism is important since it is believed to produce strong wave-particle interactions in the magnetosphere with large resultant pitch angle and energy scattering of the energetic electrons. Some examples of coherent VLF waves which are thought to be involved in this strong nonlinear interaction include natural whistlers [Helliwell et al., 1973], naturally and artificially triggered VLF emissions [Stiles and Helliwell, 1975], signals from VLF ground transmitters [Helliwell and Katsufukis, 1974], and harmonic waves from large scale power grids [Helliwell et al., 1975; Park, 1976]. An additional future source of coherent VLF waves in the magnetosphere may be VLF wave-injection experiments using satellite borne transmitters.

In our study we have focused on magnetospheric parameters appropriate for the L = 4 field line. This is the approximate location of the Siple VLF transmitter in Antarctica, the source of much of the data concerning nonlinear interactions between the coherent VLF waves and energetic electrons in the magnetosphere.

We have used a computer simulation of the full nonlinear equations of motion for energetic particles interacting with a longitudinal whistler mode wave in an inhomogeneous magnetoplasma. We have studied, in detail, interaction of single particles, sheets of particles uniformly distributed in phase and a full distribution of particles with a given wave. The interaction of a full particle distribution is studied with a
test particle approach. In this approach, in order to estimate the perturbation of the full distribution, complete trajectories of sufficient number of particles distributed appropriately in phase space are computed.

We have shown how the nonlinear pitch angle scattering varies as a function of particle pitch angle, wave amplitude, cold plasma density, and resonance position along the magnetic field line. We have compared the nonlinear theory with the well-known linear theory and have derived a convenient quantitative criterion for determining the applicability of linear theory in any particular case. In particular, our results indicate that nonlinear effects are significant for wave amplitudes as low as 3 mV for a 5 kHz signal near the magnetic equatorial plane at L = 4. Detailed study of the strong trapping and scattering case have shown that significant pitch angle changes can be induced in the energetic particle population. These results indicate that at high wave amplitudes even particles with very high pitch angles (≤ 60°) can be scattered into the loss cone in a single encounter with the wave.

Our full distribution calculations show that significant precipitated energetic electron fluxes can be produced with moderate strength VLF waves. For example our results indicate that at L = 4 a 10 mV, 5 kHz wave can produce a precipitated energy flux of 1-2 keV electrons of as much as 10^{-1} ergs/cm^2-sec, the exact value depending upon the value of the energetic electron distribution function near the loss cone. We have shown that significant leverage is involved in the wave induced particle precipitation process. Typically, the energy density of the precipitated flux is 50-60 dB higher than that of the wave. We have computed the ionospheric perturbations due to these precipitated fluxes.
and have shown that significant density enhancements can be induced in the night-time ionosphere.

In summary, we have shown that significant energetic electron pitch angle scattering and precipitation can be produced by coherent VLF waves of moderate amplitude during nonlinear cyclotron resonance interactions in inhomogeneous magnetoplasmas. This type of scattering has not previously been considered in the large volume of work which has appeared in the literature over the last decade concerning energetic particle pitch angle scattering in the magnetosphere. Although it remains to be seen what role coherent wave scattering plays in determining the large scale characteristics of energetic electrons in the magnetosphere, it is clear that this type of scattering should play an important part in VLF wave-injection experiments, both from the ground and in space.

B. MEASUREMENT OF THE PRECIPITATED FLUX

Assuming that our calculations are correct and that an energetic electron flux of $10^{-1}$ ergs/cm$^2$-sec can be precipitated into the atmosphere during VLF wave injection experiments, the question arises as to how this flux or the ionospheric perturbations produced by this flux can be detected.

The most direct method of detection is to employ satellite particle detectors. However this method has drawbacks at both high and low altitude. At high altitude near the magnetic equatorial plane the precipitating fluxes are less intense and the particle detectors must be directed into the loss cone. Since the half angle of the loss cone is very small, (-5° at L - 4) the pointing accuracy of the detector must be high and its angular resolution must be at least equal to the loss cone half
angle. At low altitude (~500 km) the loss cone half angle approaches 90° and the detector pointing accuracy and angular resolution are not critical parameters. However the satellite velocity is high (~7 km/sec) and the region of precipitation may be small (~100 km scale) so that the time available for flux measurements may be severely limited.

A second method involves the use of photometers at ground stations such as Siple Station, Antarctica. If an experiment lasts for several weeks, there is a good probability that the precipitation region will sometimes be located within 100 km of the transmitter. In this case enhanced photo emission from the atmosphere should be readily detectable on the ground near the transmitter if the precipitated flux exceeds 0.01 ergs/cm²-sec. Recent photometer measurements at Siple Station have shown that significant enhancements in the photometer output can be produced by precipitation induced by VLF whistler-mode noise bursts at 2-4 kHz [J. Doolittle, private communication].

A third method makes use of precipitation induced modification of the D region. In this effect the energetic electrons are precipitated into the atmosphere, penetrate the D layer and change the properties of the earth-ionosphere waveguide, altering their amplitude and/or phase of VLF waves propagating over long distances (~1000 km) in the waveguide. This type of modification has been observed to occur because of scattering of energetic electrons by whistlers [Helliwell et al., 1973]. It appears that the same type of interaction should take place whether the coherent input scattering wave is a signal from a VLF transmitter, a natural whistler or a discrete emission. Thus wave-induced particle precipitation events may be detectable through this effect. Our calcu-
lations indicate that the flux of electrons of energy $30 \pm 0.5$ keV precipitated by a CW input wave of 10 mV amplitude and a frequency of 5 kHz should lie in the range $10^{-2}$-$10^{-3}$ ergs/cm$^2$-sec, depending upon the value of the particle distribution function near the loss cone. This flux produces a substantial electron density enhancement below the nighttime D layer and should lead to detectable perturbations in the VLF waves propagating in the earth-ionosphere waveguide.

One other method of precipitation detection is high altitude balloon measurements of the Bremmstrahlung x-rays produced by the precipitating electrons. Rosenberg et al. [1971] have measured one-to-one correlation between VLF emissions and bursts on the balloon x-ray measurements.

Additional methods of precipitation detection include riometer and ionosonde techniques.

C. INTENSITY OF COHÉRENT WAVES IN THE MAGNETOSPHERE

Although we have presented our results for a wide range of wave intensities we have, from time to time, stressed the results for a 10 mV wave amplitude. Such a wave intensity is a reasonable estimate of the amplitude of coherent whistler mode waves in the magnetosphere. The evidence for this comes predominantly from high altitude satellite data. For instance the amplitude of the navigational VLF stations NAA (17.8 kHz) and NPG (18.6 kHz) were often measured by a VLF experiment on the high altitude satellite OGO 1 when this satellite crossed the magnetic field lines linking the transmitters ($L = 3$). It was found that 10 mV was a representative amplitude for the transmitter signals near the magnetic equatorial plane [Heyborne, 1966]. This wave amplitude was asso-
associated with a radiated power output of one megawatt for NAA and 1/4 megawatt for NPG.

The amplitude of VLF waves in the magnetosphere produced by the relatively low power (output \(\leq 1 \text{ KW}\)) Siple Station transmitter has been measured only once to date. On this occasion a wave amplitude of 0.3 my was measured on the high altitude satellite IMP-6 when at a latitude of 20°S on the field lines (\(L \sim 4\)) linking the transmitter [Inan et al., 1977a]. Since the satellite intercepted the signal on the transmitter side of the equator and before the signal has traveled once over the field line, it was concluded that this was an unamplified wave amplitude.

Considering the commonly observed 20-30 dB amplification of VLF signals in the magnetosphere [Helliwell and Katsufrikas, 1974], the signal intensity at the equator could be as high as 3-8 my. The amplitude of the naturally occurring highly coherent VLF signals known as "chorus" has been measured on a number of satellites [Burtis and Helliwell, 1976; Tsurutani and Smith, 1974; Taylor and Gurnett, 1968]. The amplitude of these quasi-coherent signals near 5 kHz has typically been found to lie in the range 2-20 my.

Furthermore, because of the high whistler mode radiation efficiency of dipole and loop antennas at VLF frequencies in the ionosphere and magnetosphere [Wang and Bell, 1972], it appears possible to produce a 10 my wave near the magnetic equatorial plane on the \(L = 3-5\) field lines using a Space-Shuttle based VLF transmitter of 1 kilowatt radiated power.

Similarly a high altitude satellite based VLF transmitter of 1 kilowatt output, which operated within a few thousand km of the magnetic equatorial plane, could be expected to produce a wave field exceeding
100 my within 1000 km of the equatorial plane.

Thus a 10 my amplitude is representative of highly coherent VLF wave types that are presently found, or that may be introduced in the future, in the magnetosphere. When these waves are present, strong pitch angle scattering of energetic electrons can be expected. Details of this scattering will depend upon the spectral form of the coherent wave. Our present results, which are based on the assumption of a fixed frequency wave, are meant to apply primarily to the case of VLF wave injection experiments involving fixed frequency inputs near the L = 4 field lines. These experiments can be either ground, space station or satellite based. However, our results can also be applied to the case of pitch angle scattering by chorus and whistler elements so long as the wave frequency changes slowly with time.

D. DISCUSSION

The particle scattering calculations presented in the main body of this report have been carried out under the assumption that the wave amplitude is reasonably constant over the region near the magnetic equatorial plane where significant scattering takes place. In other words, our calculations do not include the effects of the electromagnetic fields generated by the perturbed energetic particles. This assumption will be valid if the currents stimulated in the energetic particle population by the incident wave do not lead to significant damping or amplification of the wave near the magnetic equatorial plane. Although linear theory cannot describe the long term behavior of our system, the initial behavior of the system can be obtained from linear theory using the linear Boltzmann-Vlasov theory [Stix, 1962; Kennel and Petschek, 1966; Bernard,
1973]. It should be noted that many of the observed features of wave growth and emission generation in the magnetosphere, especially those that show generation of multiple frequencies and wave entrainment effects [Helliwell and Katsufrakis, 1974; Helliwell et al., 1975; Raghuram et al., 1977], cannot be explained by using linear theory. Our argument in this section makes use of linear theory to describe the initial behavior of the system. The following expression can be derived for the asymptotic spatial growth rate for longitudinally propagating whistler mode waves in a homogeneous relatively cold magnetic plasma with a dilute energetic electron component:

\[
k_\perp = \pi^2 \left( \frac{N_h}{N_c} \right) \left( \frac{\omega_H - \omega}{\omega} \right) \left[ 1 - \left( \frac{\omega}{\omega_H} - 1 \right) \right] A F(v_R)
\]

(6.1)

where

\[
A = F^{-1}(v_R) \int_0^\infty \tan \alpha \left| \frac{\partial f_h}{\partial \alpha} \right| \left. \frac{v_\perp}{v_\parallel = v_R} \right| dv_\perp
\]

\[
F(v_R) = \int_0^\infty f_h(v_\perp, v_\parallel = v_R) dv_\perp
\]

and \( f_h(v_\perp, v_\parallel) \) is the normalized energetic electron distribution function; \( v_R = \frac{\omega_H - \omega}{k} \) is the local resonance velocity, \( N_h \) is the total number of energetic (hot) electrons, \( N_c \) is the total number of cold electrons and it is assumed that \( N_h \ll N_c \).

Although (6.1) is derived for a homogeneous plasma, it can be expected to hold point to point in a slowly varying plasma such as that of the magnetosphere. When this is the case, the total amplitude change over a distance \( S \) along the magnetic field lines can be obtained by
the relation

\[ \bar{B}_F = \bar{B}_i e^{-\int_0^S k_\perp dz} \]  \hspace{1cm} (6.2)

where \( \bar{B}_i \) and \( \bar{B}_F \) are the initial and final wave magnetic field intensities respectively.

Schield and Frank [1970] have reported quiet time electron fluxes measured on the high altitude satellite, OGO-3, on L shells ranging from 3 to 10. During the measurements (June, July, 1966) the satellite was located near local midnight at low magnetic latitudes.

Near L = 4, the differential energy spectrum, \( \phi \), of electrons near 90° pitch angle could be closely fitted by the relation:

\[ \phi = 10^8 \left( \frac{E}{E_0} \right) \text{electrons/cm}^2\text{-steradian-keV-sec} \]  \hspace{1cm} (6.3)

where \( E_0 = 1 \text{ keV} \) and \( E \geq E_0 \). If we assume that the electrons were isotropically distributed in pitch angle, then we can use (6.1) and (6.3) in conjunction with the parameters of Table 1 to determine the initial wave attenuation rate due to the presence of these energetic electrons. We obtain the result

\[ k_\perp = -2.5 \times 10^{-3} \text{km}^{-1} \]  \hspace{1cm} (6.4)

For commonly accepted models of the cold plasma distribution along the earth's magnetic field lines it can be shown that the damping rate of (6.1) remains reasonably constant over a distance of approximately \( \pm 4000 \text{ km} \) about the magnetic equatorial plane near the L = 4 field line. In this case the linear theory predicts total damping at 5 kHz of
approximately 170 dB.

However, it can be shown [Dysthe, 1971; Palmadesso and Schmidt, 1971; Bud'ko et al., 1972] that the linear growth (damping) rate of (6.1) can apply at most over a distance $S_T = v_R \bar{T}_T$ where $\bar{T}_T$ is the average trapping time of the energetic resonant particles. Assuming that the energetic electron distribution falls off as $v^{-4}$, we find (using the parameters of Table 1) that the trapping distance has the value $S_T = 400$ km and total wave intensity change over this distance is approximately -9 dB. Thus during the time of its validity, linear theory predicts a substantial wave damping due to the quiet time energetic electron fluxes. With damping of this magnitude it can be concluded that whistlers and other natural whistler-mode signals would not be able to propagate between hemispheres. However it is just in magnetically quiet periods that whistler and VLF emission activity tends to peak [Carpenter and Miller, 1976]. Thus it must be the case that quiet time energetic electron fluxes in the range 1-10 keV are generally not isotropically distributed with respect to pitch angle. This conclusion is supported by recent results from the Explorer 45 spacecraft which show large anisotropies in energetic electron pitch angle distributions during both quiet [Lyons and Williams, 1975] and disturbed [Anderson, 1976] periods in the premidnight sector near L = 4.

In the past, a number of workers [Bell and Buneman, 1964; Kennel and Petschek, 1966; Liemohn, 1967] have attributed the amplification of whistlers and the generation of VLF emissions to pitch angle anisotropies in the energetic electron distribution function. In fact the gain predicted by (6.1) can be quite high for moderate anisotropies. For ex-
ample if the flux of Schield and Frank is assumed to have a \( \sin^2 \alpha \) dependence, (6.2) predicts a total gain in wave intensity of about 100 dB at 5 kHz near \( L \sim 4 \) (6 dB per 400 km).

Although pitch angle anisotropies may play an important role in the amplification of whistler mode signals and in the generation of VLF emissions, the probability of observing such effects during a given quiet period is only about 25% [Carpenter and Miller, 1976]. The lack of significant wave-particle interactions during the remainder of the time has been attributed by Bernard [1973] to a relaxation of the energetic electron distribution function to a state of marginal stability where, in terms of (6.3), \( A = \left( \frac{\omega_p}{\omega} - 1 \right)^{-1} \) over a wide band of wave frequency. This relaxation may take place through the Kennel-Petschek [1966] mechanism, but does not necessarily require a large decrease in flux to achieve the marginally stable state. Thus it is possible that fluxes such as those reported by Schield and Frank can exist in a state of marginal stability in which injected whistler mode waves will not exchange a significant amount of energy with the energetic electrons. However, under these conditions a significant scattering of electrons into the loss cone can still take place. Thus our calculations can be expected to apply during magnetically quiet times following magnetic disturbances when the energetic electron distribution function has relaxed to a state marginally stable to injected whistler mode waves.

Our results will also be expected to apply whenever the quiet time energetic particle fluxes are significantly less than those reported by Schield and Frank [1970]. For instance Lyons and Williams [1975] report a quiet time energetic electron flux in the premidnight sector...
which is almost two orders of magnitude lower than that reported by Frank for the midnight sector. Using the Lyons and Williams data in (6.1), we find that the assumption of isotropy leads to total attenuation of approximately 2 dB, while the assumption of a $\sin^2\alpha$ anisotropy leads to a total gain of approximately 2 dB. Clearly for these flux levels, the currents stimulated in the energetic particle distribution should have little effect on the wave amplitude. However it is also clear that the precipitated flux will be smaller in these cases where the ambient flux levels are low.

E. APPLICATIONS AND FUTURE IMPLICATIONS

One important conclusion of our study is that significant precipitated particle fluxes can be induced by coherent VLF signals from ground transmitters. The leverage involved in this interaction is very large. As shown in Chapter 5 the energy content of the precipitated flux is as much as 50-60 dB higher than that of the input wave. Thus our results indicate the possibility of controlled particle precipitation out of the radiation belts. Such controlled precipitation would have many conceivable applications, including the following:

i) Study the physics of the aurora and the lower ionosphere, by controlling x-ray production, ionization, radiation, and recombination processes, chemical and transport processes, and the coupling between the ionosphere and the magnetosphere.

ii) Control the conductivity and hence the current flow in the E and D regions of the ionosphere. It has been suggested that by varying such currents in a periodic manner ULF radiation in the micropulsation frequency range (1-10 Hz) could be produced [Bell, 1976].
iii) Control density enhancements in the D region of the ionosphere and change the properties of the earth-ionosphere waveguide, causing phase and amplitude scintillation of VLF waves in the waveguide. Thus controlled precipitation can be used to modify the D region for the benefit of VLF communication and navigation systems.

iv) The ionizing radiation from energetic particles trapped in the earth's radiation belts degrades the performance of solar cells and other sensitive equipment during high altitude space flights in the vicinity of the radiation belts. For manned missions this radiation is a serious health hazard to the crew. Controlled particle precipitation can be used to diminish the average energy in the radiation belts by reducing the number of trapped particles. This application requires relatively higher wave intensities (> 50 nT). For this reason a satellite transmitter might be more suitable.

v) We have shown that large changes (>20°) in particle pitch angle can be induced by coherent VLF waves. Such pitch angle perturbations can be measured by present satellite detectors. Thus it seems possible to use controlled perturbations of the trapped particle distribution function in the magnetosphere in order to study the physics of the radiation belt particles.

vi) The cyclotron resonance interaction that we have studied here will be present in a wide range of plasmas, both natural and manmade. Thus our results could aid in the understanding of laboratory plasmas and plasma physics in general.

vii) As a very long term application, we can speculate that controlled particle precipitation may one day be used to modify weather
processes. The precipitated flux locally heats the ionosphere and this heat may be convected down to the atmosphere thereby changing the stratospheric temperature and affecting the weather. Much more data concerning coupling between the ionosphere and the atmosphere must be acquired before the feasibility of this idea can be evaluated.

F. SUGGESTIONS FOR FUTURE WORK

We have presented a computer simulation of the complete trajectories of a distribution of radiation belt particles interacting with a given longitudinal whistler mode wave in the magnetosphere. We list below some possible extensions of this work:

i) **Iteration:** In Chapter 5, we have computed the precipitated flux from 'one pass' of the wave allowing each particle in the initial distribution to interact only once with the wave. The result is a one-pass perturbed distribution. One obvious straightforward extension is to iterate the procedure. That is, to repeat the calculations using as the initial distribution the one-pass perturbed distribution. Iterating further, the time development of the precipitated flux and the distribution function can be calculated. In order for this iteration to reach a steady state the source function must be taken into consideration. Two natural source functions are those provided by particle injection from the tail and convective injection through radial and azimuthal drifts (see section 2.B).

ii) **Non-longitudinal propagation:** In our formulation we have considered only longitudinally propagating whistler-mode waves. For waves propagating at non-zero wave normal angles both the electric and magnetic fields of the wave have components along the static magnetic field.
In addition, the transverse polarization of the wave is elliptical rather than circular. Therefore the equations of motion have additional terms. Also, a non-longitudinal wave does not follow a given field line and its path in the magnetosphere must be calculated by using ray tracing. The results of the ray tracing must then be incorporated into the simulation in order to obtain wave and medium parameters such as wave number, wave normal and gyrofrequency along the ray path. In addition to first order cyclotron resonance as described by (2.30) a non-longitudinal wave resonates with the particles in the Landau, or longitudinal resonance mode when $v_p = v_\| \cos \theta$. Since there is a non-zero parallel component of the electric field, energy exchange between the wave and the particle can occur in this case. Furthermore there is space charge bunching, again as a result of the parallel electric field component. A given non-longitudinal wave will resonate with the particles in both the cyclotron and Landau mode, as well as all other harmonic resonances, $\omega + k v_\| = m\omega_H$, $m = 0, \pm 1, \pm 2$, etc. Fortunately, in the magnetosphere the cyclotron and Landau resonance interactions are fully separable, since the former occurs when the wave and the particles travel in opposite directions whereas the latter occurs when they travel in the same direction.

The simulation of the cyclotron resonance interaction with quasi-longitudinal waves, i.e., waves which propagate more or less along the magnetic field line but with $\theta < 30^\circ$, can be simulated with some modification to our computer program. In this case the wave path is defined and the major changes occur only in the equations of motion. The simulation of the Landau resonance interaction between a quasi-
longitudinal wave and the particles requires little modification in the
program. The adiabatic dynamics of particles, the manipulation of the
distribution function and the computation of the medium parameters need
not be changed. However one must use a different set equations of
motions.

iii) Variation of the wave frequency: Our present program uses mono-
chromatic pulses as the wave function. Most of the ground transmitter
signals, although not all, and the signals induced by large scale power
grids are in this form [Helliwell and Katsufrakis, 1974; Helliwell et al.,
1975]. However many interesting wave particle or wave-particle-wave
interaction phenomena seen in VLF data involve one or more waves at
different constant frequencies or one or more waves with varying fre-
quency. To study these phenomena quantitatively it is necessary to mod-
ify the program to cover the more general case of multi-component waves
with varying frequency. The required changes in the program lie in the
computation of a wave structure and appear to be straightforward in
nature. One other important application of a program using a variable
frequency wave would be to determine the optimum wave frequency vari-
tion which would extend the resonance time and optimize the precipitated
flux. To first order, for a given wave intensity the wave frequency must
vary such that Eq. (2.30) is satisfied over a longer distance along the
field line. However the wave induced changes in particle parallel
velocity complicate this picture. If $B_w$ is large enough (2.30) can be
satisfied over a long path for even a fixed frequency wave.

iv) Feedback: The most general solution to the problem of gyro-
resonant wave-particle interaction in the magnetosphere should include
the amplification or growth of the wave, i.e. the energy transfer from the particles to the wave.

In our studies up to this point we have assumed a wave structure and computed the particle mechanics. This is not a very limiting assumption since one could use a spatially growing wave as the wave structure and hence indirectly account for the effect of the particles on the wave. The spatial growth pattern could be calculated in a separate analysis or deduced from data.

The full self-consistent solution however would be a computation which modifies the wave in accordance with the fields radiated by the currents produced by the phase bunched particles [Helliwell and Crystal, 1973] while they are scattered by the wave.

It should be noted that in our present program the full phase motion of all particles, which in effect gives the phase bunched currents, is already computed. Therefore it is conceivable that the present program can be used to find the new fields radiated by the particles. However, this must be done at each step of the interaction on a space-time frame and the computation becomes complicated. Although we believe that this feedback computation could be done, with some limitations, we think that it requires important modifications of the present program.
APPENDIX A: SHAPE OF $\Delta \alpha_{eq}$ vs. $\phi_0$ CURVES

In this appendix we will present a semi-quantitative analysis aimed at understanding the shapes of the $\Delta \alpha_{eq}$ vs. $\phi_0$ curves discussed in Chapter 4. We first show that for linear theory the total scattering is proportional to $\sin \phi_0$.

By definition, linear theory assumes that the particle scattering can be computed using the unperturbed phase motion of the particle. This is explained in section 2.5 where we discuss the linear theory and have derived expressions for $\Delta \nu_\parallel$ and $\Delta \nu_\perp$. Note that for the single equatorially resonant sheet which we have considered in Figs. 4.2 through 4.9 all initial phases have the same $\psi_0$ and $\alpha_0$, and therefore the variation $\phi = f(t)$ as given by (2.51) is the same for all initial phases. Rewriting the expression (2.53a) for $\Delta \nu_\parallel$ we have

$$\Delta \nu_\parallel = \left( -\frac{eB}{m} \right) \nu_\perp \int_0^T \sin(F(t) + \phi_0) \, dt \quad (A.1)$$

Using trigonometric identities, (A.1) can be written as

$$\Delta \nu_\parallel = \frac{eB}{m} \nu_\perp \left[ \int_0^T \sin \phi_0 \cos[F(t)] \, dt + \int_0^T \cos \phi_0 \sin[F(t)] \, dt \right] \quad (A.2)$$

Since $F(t) = \int_0^t f(t') \, dt'$ is independent of $\phi_0$, so are $\int_0^T \cos[F(t)] \, dt$ and $\int_0^T \sin[F(t)] \, dt$. Let

$$K_1 = \int_0^T \cos[F(t)] \, dt, \quad K_2 = \int_0^T \sin[F(t)] \, dt \quad (A.3)$$
\[
\Delta v_\parallel = \frac{eB}{m} v_\parallel \left[ K_1 \sin \phi_0 + K_2 \cos \phi_0 \right] = \frac{eB}{m} v_\parallel \frac{\sqrt{K_1^2 + K_2^2}}{K_1} \sin \left( \phi_0 + \beta \right) \quad (A.4)
\]

where \( \beta = \tan^{-1} \frac{K_2}{K_1} \).

Similarly by manipulating Eq. (2.53b) we obtain:

\[
\Delta v_\perp = -\frac{eB}{m} \left( v_\parallel + \frac{\omega}{k} \right) \frac{\sqrt{K_1^2 + K_2^2}}{K_2} \sin(\phi_0 + \beta) \quad (A.5)
\]

Now since \( \alpha = \tan^{-1} \frac{v_\perp}{v_\parallel} \) we have:

\[
\Delta \alpha = \cos^2 \alpha \left( \frac{1}{v_\parallel} - \frac{v_\perp}{v_\parallel^2} \frac{\Delta v_\parallel}{\Delta v_\perp} \right) \Delta v_\perp \quad (A.6)
\]

By substituting (A.4) and (A.5) into (A.6) we obtain

\[
\Delta \alpha = C(v_\parallel, \alpha_0) \sin(\phi_0 + \beta) \quad (A.7)
\]

where \( C(v_\parallel, \alpha_0) \) is independent of \( \phi_0 \) and \( \beta \). Equation (A.7) shows that \( \Delta \alpha \) vs. \( \phi_0 \) curves obtained using linear theory should have a sinusoidal shape as indicated by Fig. 4.5. For \( \rho < 1 \), the full solution of Eqs. (2.36) is closely approximated by the linear theory, hence the \( \Delta \alpha \) vs. \( \phi_0 \) curves are almost sinusoidal.

For \( \rho > 1 \), the linear approximation is no longer valid. In other words, the phase variation for different \( \phi_0 \) is not the same, since \( \dot{v}_\parallel \) is different for different \( \phi_0 \). Moreover, the interaction time \( T \) is also a function of \( \phi_0 \). Therefore Eq. (A.2) cannot be further simplified and the total scattering \( \Delta v_\parallel \) for each \( \phi_0 \) must be found by a
general solution of Eqs. (2.36).

Although it is hardly possible to do this analytically, the shape of the $\Delta \alpha_{eq}$ vs. $\phi_0$ curves for the nonlinear case can be understood qualitatively by referring to the discussion in connection with Figs. 4.2 and 4.3. The basic idea is that for $\rho >> 1$, particles with initial phases around $\phi_0 = 0$ are trapped in the wave. For the trapped particles $v_\parallel = v_R$ and $\dot{v}_\parallel = \dot{v}_R$ since $\dot{v}_R > 0$ for particles moving southward from the equator, $\dot{v}_\parallel > 0$ and from Eqs. (4a,b) $\dot{v}_\perp < 0$. Thus $\Delta \alpha_{eq} < 0$ for these trapped particles.
APPENDIX B: LOSS CONE REFLECTION EFFECT

This appendix briefly describes what we term the "loss cone reflection effect" to which we have attributed some aspects of Figs. 4.4, 4.14 and 4.15. This effect becomes prominent for very low pitch angles and/or high wave intensities. It is caused by that term in Eq. (2.36c) which is proportional to \( B \frac{\cos \phi}{v_\perp} \).

In all our computer calculations we have kept this term in the equations of motion. Therefore its effects are seen whenever \( v_\perp \) is low enough, i.e. the pitch angle is below a few degrees. In our qualitative discussions however, especially those in connection with the 'pendulum' equation (2.45) we have ignored this term. In the following we reexamine the phase variation to clearly see the effect of this term.

Assume that \( B \frac{\cos \phi}{v_\perp} \) is so large that \( \dot{\phi} \) is approximately given by

\[
\dot{\phi} = - \left( \frac{eB}{m} \right) \left( v_\parallel + \frac{\omega}{k} \right) \cos \phi \frac{v_\perp}{v_\perp} \quad \text{(B.1)}
\]

Rewriting Eq. (2.36b) without the adiabatic term we have

\[
\dot{v}_\perp = - \left( \frac{eB}{m} \right) (v_\parallel + \frac{\omega}{k}) \sin \phi \quad \text{(B.2)}
\]

Dividing Eq. (B.2) by (B.1) we obtain

\[
\frac{dv_\perp}{v_\perp} = \frac{\sin \phi d\phi}{\cos \phi} \quad \text{(B.3)}
\]

or, integrating,

\[
\ln(v_\perp) = -\ln(\cos \phi) + \text{constant} \quad \text{(B.4)}
\]
or

\[ v_\perp \cos \phi = \text{constant} \]  \hspace{1cm} (B.5)

Eq. (B.5) shows that for the cases when \( \phi \) can be approximated by (B.1), \( v_\perp \cos \phi \) is a constant of the motion. This constant of motion is a reduced form of the general one that gives the conservation of angular momentum [Bell, 1964].

Note that during the course of any interaction, especially in the trapped mode as described in section 4.C, in which the pitch angle (therefore \( v_\perp \)) continuously decreases there will always come a time where \( v_\perp \) is so low that (B.1) would be true. Assume this occurs at time \( t_a \) when \( v_\perp = v_{\perp a} \) and \( \phi = \phi_a \). Then we have

\[ v_\perp \cos \phi = v_{\perp a} \cos \phi_a \]  \hspace{1cm} (B.6)

since \( |\cos \phi| < 1 \), it is apparent that \( v_\perp \) is bounded on the lower end so that it cannot decrease below a value \( v_{\perp a} \cos \phi_a \). Therefore the effect of the \( \frac{\cos \phi}{w} \frac{\dot{v}_\perp}{v_\perp} \) term in Eq. (2.36c) is to effectively prevent the pitch angle from reaching zero. Since \( \phi \) continuously changes (\( \dot{\phi}(t) \neq 0 \)) \( v_\perp \) has to change also and since it is bounded on the lower end, it increases. This can be seen by considering a worst case.

Assume at some \( t > t_a \), \( \phi = 0 \). In that case \( v_\perp = v_{\perp a} \cos \phi_a \) and \( \dot{v}_\perp = 0 \). But \( \dot{\phi} \) is a maximum. Therefore \( \phi \) goes negative and \( v_\perp \) has to increase through (B.2 and B.5). Note that once \( \dot{v}_\perp \) becomes positive and \( \dot{v}_\perp \) starts to increase this trend cannot be reversed. As \( \dot{\phi} \) approaches \( -\pi/2 \), \( v_\perp \) increases through (B.6) and eventually the approximation (B.1) breaks down and the first term in Eq. (2.36c) takes over.
The conclusion of the above analysis is the following: if at any time during the interaction \( \phi \) approaches zero closely enough so that (B.1) becomes valid, it will be 'reflected' back and increase to a value such that (B.1) is not valid. This reflection is termed the 'loss cone reflection effect'. An example of this effect is seen in the sharp reflection of pitch angle at the second minimum of the \( \Delta \alpha_{eq}(t) \) for \( \phi_0 = -\pi/3 \) particle the trajectory of which is given in Fig. 4.3. At that point \( \Delta \alpha_{eq} = \alpha_{eq0} \).

The slow decrease of \( \Delta \alpha_{eq} \) for the \( \alpha_{eq0} = 10^\circ \) and \( 30^\circ \) particles for \( B_w \geq 50 \text{ mT} \) and \( 100 \text{ mT} \) are also due partially to this effect. However these cases are more difficult to describe and will not be detailed here.

Note that in addition to the loss cone reflection effect the \( B_w \frac{\cos \phi}{v_\perp} \) term also changes the mode of trapping when \( B_w \) is sufficiently large. This occurs when \( |w_H - w_\parallel| \approx |(\frac{eB_w}{m})(\gamma_\parallel + \frac{\omega_\parallel}{k}) \cos \phi| \) so that neither (B.1) nor (2.38) is not valid. In that case one has to use the complete equation (2.36c) for \( \dot{\phi} \). The interaction is no longer analogous to a simple pendulum with an applied torque.
APPENDIX C: JACOBIAN FOR $f(v_\parallel, \alpha, \phi)$

In this appendix we demonstrate that the velocity space volume element in terms of $v_\parallel$ and $\alpha$ is $v_\parallel^2 \frac{\sin \alpha}{\cos^3 \alpha} d\alpha dv_\parallel d\phi$ as used in Chapter 5. The coordinate system is shown below.

\[ (v_\parallel, v_\perp, \alpha, \phi) \]

\[ (v_x, v_y, v_z) \]

We have

\[ v_x = v \sin \alpha \cos \phi = v_\parallel \frac{\sin \alpha}{\cos \alpha} \cos \phi \]

\[ v_y = v \sin \alpha \sin \phi = v_\parallel \frac{\sin \alpha}{\cos \alpha} \sin \phi \]

\[ v_z = v \cos \alpha = v_\parallel \]

The Jacobian is then given by,
\[ J = \frac{\partial (v_x, v_y, v_z)}{\partial (v_\parallel, \alpha, \phi)} = \begin{vmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial \alpha} & \frac{\partial v_x}{\partial \phi} \\ \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial \alpha} & \frac{\partial v_y}{\partial \phi} \\ \frac{\partial v_z}{\partial z} & \frac{\partial v_z}{\partial \alpha} & \frac{\partial v_z}{\partial \phi} \end{vmatrix} \]

\[ = \begin{vmatrix} \frac{\sin \alpha}{\cos \alpha} \cos \phi & v_\parallel \frac{1}{\cos^2 \alpha} \cos \phi & -v_\parallel \frac{\sin \alpha}{\cos \alpha} \sin \phi \\ \frac{\sin \alpha}{\cos \alpha} \sin \phi & v_\parallel \frac{1}{\cos^2 \alpha} \sin \phi & v_\parallel \frac{\sin \alpha}{\cos \alpha} \cos \phi \\ 1 & 0 & 0 \end{vmatrix} \]

\[ = - (v_\parallel \frac{1}{\cos^2 \alpha} \cos \phi)(-v_\parallel \frac{\sin \alpha}{\cos \alpha} \cos \phi) + (-v_\parallel \frac{\sin \alpha}{\cos \alpha} \sin \phi)(-v_\parallel \frac{1}{\cos^2 \alpha} \sin \phi) \]

\[ = v_\parallel^2 \frac{\sin \alpha}{\cos^3 \alpha} [\cos^2 \phi + \sin^2 \phi] = v_\parallel^2 \frac{\sin \alpha}{\cos^3 \alpha} \]

Hence the volume element in the \( v_\parallel, \alpha, \phi \) coordinate system is

\[ dV = v_\parallel^2 \frac{\sin \alpha}{\cos^3 \alpha} \, d\alpha \, dv_\parallel \, d\phi \]
APPENDIX D: THE COMPUTER SIMULATION

This program is a simplified version of the one used for the computations in this report.

The program simulates the cyclotron resonance interaction of energetic electrons with a fixed frequency wave of constant amplitude in the Magnetosphere.

In the following we give a listing of the program together with brief descriptions of various sections.

INPUT PARAMETERS

\[ F = \text{wave frequency (degrees)} \]
\[ YL = \text{L value of the field line} \]
\[ PHILD = \text{Latitude at 1000 km altitude (degrees)} \]
\[ YNEQ = \text{Equatorial cold plasma density (el/cc)} \]
\[ YLAMAX = \text{Latitude range (see Chapter III) (Degrees)} \]
\[ M = \text{Number of mesh points between - YLAMAX and Equator.} \]
\[ N = \text{Number of electrons per sheet} \]
\[ BWEQ = \text{Wave intensity (in milligammas)} \]
\[ PHIWH = \text{Initial phase of the first electron in the sheet (degrees)} \]

ALPHMN, HALPHA, ALPHMX determine the initial equatorial pitch angle for which the calculation is to be made.

Initial pitch angles varying from ALPHMN to ALPHMX in steps of HALPHA are simulated. All three quantities are in degrees.
MSTEP = Number of steps between each spatial step for which the medium is stored.

DUR = The integration is stopped when the particle is away from resonance by an amount 100* DUR percent

PHSTOP = The interaction is not stopped until DPHI becomes larger than PHSTOP (degrees).

STRT = The particle is introduced to the wave at a location where it is away from resonance by an (1-STRT)* 100 percent.

ALPHAV = Average pitch angle of the particles to be simulated. This is useful for the setting up of the parallel velocity mesh of the distribution function (Degrees).

RSTSZ = The parallel velocity mesh of the distribution function is set up in such a way that the mesh points are separated in the corresponding resonance latitude by RSTSZ (Degrees).

NPITCH = Pitch angle mesh size for the distribution.

NVB = Number of mesh points in parallel velocity.

NPB = Number of mesh points in pitch angle.

DSTPLT = If DSTPLT is less than 1.0 the full distribution computations not made.

PLT = If PLT is less than 1.0 the single particle trajectories are not plotted.

RANGE = Pitch angle range (degrees) for the plot of single particle trajectory.

RANGPH = Phase range (degrees) for the same plot.

IGRSTP = Number of integration steps between each plotted point.
YLF, YTT, YLST determine the initial equatorial parallel velocity for which the calculation is to be made. Parallel velocities are determined according to latitude of resonance. Particles with equatorial parallel velocities such that they will resonate with the wave at latitudes YLF to YTT in steps of YLST are simulated. All three quantities are in degrees.
C
THE COMPUTATION OF THE MEDIUM
C

46  C1=1.0/IL
47  C2=2.0/IL (C1)
48  PHIDB=ADD (C2)
49  C3=(0.5)*(IL/1.73205*C1)
50  C4=PHIDB*C1*CT1=6.28315*52
51  C5=CT1 (PHIDB)
52  C6=SQRT(3.0*C5+C0+1.0)
53  S0=C3*ADD (1.73205*C3+C0) + 1.73205*C5*C6
54  PHIDB=2.0*PHIDB/IL (0.0)
55  Y1=S0*1.0
56  C7=C5+S0 (PHIDB)
57  C8=C4+C7*C7
\[ ZEQ = B1 \cdot B1g / \pi \cdot EW - 3.6094e-10 \cdot ( \pi \cdot EW - C3 ) \]

\[ C9 = XEW / SQRT(0.9 \cdot EXP(-ZEQ/H1) + 0.02 \cdot EXP(-ZEQ/H2) + 0.08 \cdot EXP(-ZEQ/H3)) \]

\[ C10 = 5C / C1 \]

\[ YLAMAD = YLAMAD / PI \h
\]

\[ H = -YLAMAD / YM \]

\[ KL = M + 1 \]

\[ LO = 22 \quad I = 1, KL \]

\[ YI = I - 1 \]

\[ PHI = YLAMAD + YI \cdot H \]

\[ IF(I.EQ.1) \cdot PHI = 0.0 \]

\[ C11 = SIN(- \cdot PHI) \]

\[ C12 = 1.73205 \cdot C11 \]

\[ C13 = SQRT(3.0 \cdot C11 + C11 + 1.0) \]

\[ S = C3 \cdot (ALOG(C12 + C13) + C12 + C13) \]

\[ C14 = COS(PI / 2) \]

\[ C15 = C14 \cdot C14 \]

\[ C16 = C15 \cdot C15 \cdot C15 \]

\[ WH(I) = (C4 / C10) \cdot C13 \]

\[ Z(I) = (50 - S) \cdot 1.0 \cdot 0.3 \]

\[ CE = C10 \cdot C15 \]

\[ ZE = B1 \cdot 1.15 / \pi = 3.6094e-10 \cdot ( \pi \cdot CE - C8) \]

\[ YN = C9 \cdot SQRT(0.9 \cdot EXP(-ZEQ/H1) + 0.02 \cdot EXP(-ZEQ/H2) + 0.08 \cdot EXP(-ZEQ/H3)) \]

\[ C17 = WH(I) - W \]

\[ YK = 0.33333 \cdot 0.5 \cdot SQRT(W / C17) \cdot 1.0 \cdot 0.3 \cdot SQRT(YN \cdot 0.31333 \cdot 10) \]

\[ VPAR(I) = C17 / YK \]

\[ IF(I.EQ.1) \quad GO \quad TO \quad 22 \]

\[ 20 \quad IF(PHI.EQ.0.0) \quad GO \quad TO \quad 21 \]

\[ 22 \quad CONTINUE \]

\[ 21 \quad KL = I \]

\[ NT = 2 \cdot KL - 1 \]

\[ ERMAX = 2.64217 \cdot KEX = 15 \cdot VPAR(I) \cdot VPAR(I) \]

\[ IEQ(I) = I + (NT - 1) / 2 \]

---

**END OF COMPUTATION OF THE MEDIUM.**

**THE OUTPUT OF THIS SECTION IS THE**

**TAIRLETS 2, WH(2) AND VPAR(5) = LOCAL RESONANCE VELOCITY**

**C**

**IBF = 0**

**IBF = 0**

**MWII = BRT TSL / (PI * H)**

**C**

**SETTING UP THE PITCH ANGLE AND PARALLEL VELOCITY**

**C**

**LABELS OF THE HESHS OF THE DISTRIBUTION FUNCTION.**

**C**

**NVH MESH POINTS WITH PARALLEL VELOCITIES VI1E(1) =**

**C**

**AND NFB MESH POINTS WITH PITCH ANGLES NCHEQ(1) =**

**C**

\[ DO \quad 621 \quad KSET = 1, 3 \]

\[ YKSET = KSET - 1 \]

621 \[ PCHQK(QSET = YKSET * HPITCH * PI) \]

620 \[ DO \quad 620 \quad KSET = 2, 15 \]

620 \[ YKSET = KSET \]

620 \[ VI1E(KSET) = VPAR(IW(I)) \cdot (1.0 - 0.001 \cdot (-YKSET + 16)) \]

99 \[ IF(KSET.EQ.2) \quad GO \quad TO \quad 620 \]

100 \[ VI1EQ(KSET) = (VI1E(1) + VI1E(KSET - 1)) / 2.0 \]

100 \[ 620 \] \[ CONTINUE \]

102 \[ SNB = EQ = SIN(ALPHA) \cdot SIN(ALPHA) \]

103 \[ LQ = 622 \quad KSET = 16, NVH \]

104 \[ ILOC = IEQ + (KSET - 16) \cdot NVH \]

105 \[ ILFA = -ILC + N + 1 \]

106 \[ IF(ILOC.GT.0) \quad GO \quad TO \quad 761 \]

107 \[ YI = ILFA - 1 \]

108 \[ YLATI = YLAMAX + YI * PI \]

109 \[ IF(ILFA.EQ.ILW) \quad YLATI = 0.0 \]
ALAT(KSET) = YLAT
SA = SLC = (1.118) * S1(0) * IEQ
VIIIEQ(KSET) = SUTF(1.6 - 0.6) * VPAE(1.1P6) / 1.1 - S5.5LC)
VIIIEQ(KSET) = (VIIIEQ(KSET) + VIIIEQ(KSET - 1)) / 2.0
CONTINUE
761 VIIIEQ(1) = 0.5 * VPAE(IEQ)
VIIIEQ(1) = 0.5 * VPAE(IEQ)
IVSH = KSET - 1
IVDBP = NB + 1
VIIIEQ(2) = 2.0 - VIIIEQ(2) - VIIIEQ(3)
VIIIEQ(4) = VPAE(1) + 1.5
VIIIEQ(4) = VPAE(1)
KSET1 = KSET - 2
DO 760 KS = 16, KSET1
IE = 1.2, GE, ALAT(KS), AND, YIF, LT, ALAT(KS + 1) IBF = KS
C760 = YL, ALAT(KS)
C761 = ALAT(KS + 1) - YLT
IF (KSET, EQ, KS, AND, C760, .LT, C761) IBF = KS + 1
IF (YT, GE, ALAT(KS), AND, YIL, LT, ALAT(KS + 1)) IBT = KS
C762 = YJT - ALAT(KS)
C763 = ALAT(KS + 1) - YIT
IF (IBT, EQ, KS, AND, C762, GT, C763) IBT = KS + 1
CONTINUE
IF (IBF, EQ, 0) IBF = KS - 1
IF (IBF, EQ, 0) IBF = KS - 1
STEP = YLSI / (FRIK * H)
IBST = STEP / MVII
C
C INITIALIZATION OF THE DISTRIBUTION FUNCTION.
C IN THIS CASE FFEQ = A ABOVE THE LOSS CONE AND
C IS ZERO BELOW THE LOSS CONE.
C
DO 505 JBEQ = 1, 11, 11
503 FFEQ(IBEQ, JBEQ) = 0
DO 502 JBEQ = 12, 39
502 FFEQ(IBEQ, JBEQ) = 1
DO 505 JBEQ = 139, 505
505 FFEQ(NVB, JBEQ) = 0
CONTINUE
DO 701 JBEQ = JBF, JBT, JBST
ALPHEQ = HCPITCH * (2.0 - JBEQ - 1.0) * 0.5
INAL = TAN(ALPHEQ)
IOLD = IEQ
DO 702 IEQ = 1, 1, NBEQ
VPEEQ = VIIIEQ(IEQ)
VPEEQ = VPEEQ * INAL
VPEEQ = VPEEQ * YPEEQ
VPEEQ = VPEEQ * FPEEQ
VPEEQ = VPEEQ * FPEEQ
VPEEQ = VPEEQ * FPEEQ
IEPQ = IEQ + JBEQ + 1
VPEEQ = WH(IPFP) * FPEEQ / WH(IEQ)
VPEEQ = WH(IPFP) - YPEEQ
VPALE = QAT(VPAEQ)
C
IF (YPALE .LT. (VPAE(IBF) * STET)) GO TO 402
CONTINUE
401 CONTINUE
IF (IBF .LE. (HT + 1)) IBF = IBF
J = J - 1
VPA = VPALE
168 \text{VPEO} = \sqrt{\text{ETVPEQ}}
169 \text{IPRIME} = 1
170 \text{IF}(\text{LH}, \text{LH}) \text{IPRIME} = -I + T + 1
171 \text{ZWAT} = 1
172 \text{DO} 52 \quad \text{J} = 1, N
173 \text{LPOLE} = 0
174 \text{ICO} = \text{HZ}
175 \text{IF}(\text{DSPLT} \leq 1.0) \text{GO TO 709}
176 \text{CSA} = \text{COS} (\text{ALPHA})
177 \text{OLHFC} = \text{VPA} * \text{E} \times \text{SOQ} * \text{SIN} (\text{ALPHEQ}) / \text{CSA} * \text{CSA} * \text{CSA}
178 \text{FFEY} (\text{LBEW}, \text{JBEW}) = \text{CFF} (\text{LBEW}, \text{JBEW}) - 1.0
179 \text{GO TO 789}

\text{C PLOTTING ROUTINE}

180 \text{IF}(\text{PLT} \leq 1.0) \text{GO TO 902}
181 \text{VDF} = (\text{VPA} - \text{VPA} (\text{IPRIME})) / \text{VPA} (\text{IPRIME})
182 \text{DO} 250 \quad \text{LHA} = 1, 7
183 \text{YLHA} = \text{LHA}
184 \text{HR2} (\text{LHA}) = \text{HR} * \text{YLHA} - \text{RANGE}
185 \text{HRPH} (\text{LHA}) = (\text{HRPH} * \text{YLHA}) - \text{HANGPH}
186 \text{BEA} (\text{LHA}) = 10 * 1
187 \text{SRE} = 250 \\
188 \text{WRITE} (6,249) \text{BEWQ}, \text{VDF}
189 \text{249 FORMAT (11L, 'Bwave AT EQUATOR = ', F8.2, ' MILLIGAMMAS', ', ', VDF = ', 1E10.5)}
190 \text{WRITE} (6,251) \text{HR2P} (\text{LHA}), \text{LHA} = 1, 7, G
191 \text{251 FORMAT (3X,'****** PHASE (ANGLE BETWEEN -Bwave & VPERP) (DEGREES)'), 1/4} \\
192 / 7 (F6.2, 2X) / 39X, 79A1/)
193 \text{WRITE} (6,252) \text{HRPH} (\text{LHA}), \text{LHA} = 1, 7, G
194 \text{252 FORMAT (3X,'****** CHANGE IN EQUATORIAL PITCH ANGLE (DEGREES) '/ 1} \\
195 \text{PITCH OFF SETS, PHASE LATITUDE TIME'), 8X, 7 (F6.2, 4X) / 1X, 1(DG) \\
196 \text{2X, 1(deg) (DEG), (HSEC), 79A1)}
197 \text{DO} 254 \quad \text{LHA} = 1, 79
198 \text{GR} (\text{LHA}) = \text{HORBAR}
199 \text{WRITE} (6,255) \text{GR}
200 \text{255 FORMAT (3X, 79A1)}
201 \text{DO} 256 \quad \text{LHA} = 1, 79
202 \text{GR} (\text{LHA}) = \text{BLANK}
203 \text{GO TO 502}
204 \text{502 \text{CONTINUE}}

\text{C}

\text{VFA} = \text{VPA0}
207 \text{VPF} = \text{VPF0}
208 \text{T} = 0.0
209 \text{YJ} = 0.0
210 \text{PHI} = \text{PHIW} / \text{FH} + \text{YJ} * \text{PI} / \text{YK}
211 \text{DVF} = 0.0
212 \text{LICL} = 1
213 \text{405 IM} = 1 - 1

\text{C INTEGRATION OF EQUATIONS OF MOTION. A DIRECT SIMULATION}

\text{C OF THE EQUATIONS IS USED.}

214 \text{DO} 53 \quad \text{L} = \text{L0LD}, \text{IM} = 1
215 \text{LPH} = \text{LPH} + 1
216 \text{L} = \text{L0LQ}
217 \text{L1R} = \text{L} - 1.0 / \text{L0SST} + 1
218 \text{IF}(\text{L1RLT} \leq 1.0) \text{GO TO 903}
219 \text{ALPH} (\text{L1RL}) = \text{ATAN} (\text{VPE} / \text{VFA})

- 171 -
220 \text{SNAL} = \text{SIN} (\text{ALPH} (\text{L1} F2))
221 \text{SNAL} = \text{SIN} (\text{ALPH} (\text{L1} F2) - \text{ASIN} (\text{SIN}((\text{WH} (\text{IEQ}) - \text{WH} (\text{L}))) \times \text{SNAL} \times \text{SNAL}) - \text{ALPH})
222 \text{PHASE} (\text{L1} F2) = \text{PI} \times \text{PHI}
223 \text{YI} = -1
224 \text{YLTI} = \text{YMAX} + \text{YI} \times \text{YPIR}
225 \text{IF} (\text{L} . \text{EQ} . \text{IEQ}) \text{YLTI} = 0.0
226 \text{YL} (\text{L1} F2) = \text{YLTI}
227 \text{IF} (\text{LPR} . \text{GT} . \text{IEQ}) \text{YL} (\text{L1} F2) = -\text{YLTI}
228 \text{VPAR} (\text{L1} F2) = \text{VPE}
229 \text{VPAR} (\text{L1} F2) = \text{VPA} \times 1.0 \times 0.3
230 \text{TIM} (\text{L1} F2) = 4 \times 1000.0
231 \text{VDIF} (\text{L1} F2) = \text{VDIF} \times 100.0 / \text{VPA} (\text{L})
232 \text{L1} \text{F2} \text{LD} = \text{L1} F2
233 \text{999 CONTINUE}
234 \text{903 CONTINUE}
235 \text{CWH} = (\text{WH} (\text{L}) - \text{WH} (\text{L})) / \text{YSTEP}
236 \text{LO} 55 \text{HD} = 1, \text{NSTEP}
237 \text{VPAR} (\text{L}) = \text{VPE} \times ((\text{VPA} (\text{L})) - 1) / \text{YSTEP} \times (\text{HD} - 1)
238 \text{WH} (\text{WH} (\text{L}) + \text{CWH} \times \text{WH} (\text{L})) / \text{WH} (\text{L})
239 \text{WH} (\text{WH} (\text{L}) + \text{CWH} \times \text{WH} (\text{L})) / \text{WH} (\text{L})
240 \text{DZ} = (\text{Z} (\text{L}) - 1) / \text{YSTEP}
241 \text{IF} (\text{LPR} . \text{GT} . \text{IEQ}) \text{DZ} = -\text{DZ}
242 \text{SNPHI} = \text{SIN} (\text{PHI})
243 \text{CSPHI} = \text{COS} (\text{PHI})
244 \text{YKNA} = (\text{WH} - \text{WH} (\text{L})) / \text{VPY}
245 \text{C36} = 1.0 / \text{VPA}
246 \text{DT} = \text{DZ} \times \text{C36}
247 \text{C30} = \text{VPA} \times \text{YKNA}
248 \text{C34} = \text{VP} \times 0.5 \times (\text{WH} - \text{WH} (\text{L})) \times \text{C36} / \text{WH} (\text{L})
249 \text{C35} = \text{WH} \times \text{SNPHI} \times \text{WH}
250 \text{DVP} = -2 \times \text{C30} \times \text{C35} \times \text{C34} \times \text{VPA}
251 \text{DVP} = (\text{C35} + \text{C34}) \times \text{VP}
252 \text{VDIF} = \text{VPA} \times \text{VPA}
253 \text{DPH} = \text{DPH} = \text{DT}
254 \text{57 DPHI} = \text{DPH} \times \text{DT}
255 \text{56 CONTINUE}
256 \text{T} = \text{T} + \text{DT}
257 \text{VPE} = \text{VFP} \times \text{DVP}
258 \text{VPA} = \text{VPA} \times \text{DVP}
259 \text{VPE} = \text{VPE} \times \text{DVP}
260 \text{VPE} = \text{VPE} \times \text{DVP}
261 \text{55 CONTINUE}
262 \text{IF} (\text{DEHI} \times \text{LT} . \text{PHSTOP} . \text{AND} . \text{DPHI} . \text{GT} . \text{PHSIN}) \text{GO TO} 53
263 \text{WHAT} = \text{VPA}
264 \text{IF} (\text{WH} . \text{LT} . (\text{DUR} \times \text{VPA}) (\text{L})) \text{GO TO} 54
265 \text{IF} (\text{WH} . \text{GT} . (\text{DUR} \times \text{VPA}) (\text{L})) \text{GO TO} 54
266 \text{53 CONTINUE}
267 \text{54 IF} (\text{LPR} . \text{LT} . \text{IEQ}) \text{GO TO} 404
268 \text{55 CONTINUE}
269 \text{54 IF} (\text{LPR} . \text{LT} . \text{IEQ}) \text{GO TO} 405
270 \text{GO TO} 405
271 \text{404 CONTINUE}
272 \text{END OF INTEGRATION. THE OUTPUT OF THIS SECTION IS THE}
273 \text{LOCAL VALUES VPE AND VPA (THE PERPENDICULAR AND PARALLEL}
274 \text{VELOCITIES).}
275 \text{CONVERSION BACK TO EQUATORIAL VALUES AND ADJUSTMENT OF THE}
276 \text{DISTRIBUTION FUNCTION.}
277 \text{VPESQ} = \text{VPE} \times \text{VPE}
278 \text{GUTESQ} = \text{VPE} \times \text{VPA} \times \text{VPA}
279 \text{VPESQ} = \text{WH} (\text{IEQ}) \times \text{VPE} \times \text{WH} (\text{L})
VPEQ = QUIET - VPEQ
VPAQ = SQAT(VPEQ)
VPEQ = SQAT(VPEQ)
ALL = TAN(VPEQ/VPAQ)
DO 23 ISQAT = 1:KVB
IF(VPAQ .LE. VIIBEQ(ISQAT+1) .AND. VPAQ GT VIIBEQ(ISQAT)) GO TO 624
623 CONTINUE
624 JBEW = ISQAT
JB = ALP + M + N PITCH + 1
IF(JBEW .LT. JBEW) JBEW = 1
IF (JBEW .GT. NPB) JBEW = NPB
IR(ISQPL+1.0) GO TO 790
CSA = CGS(ALPHAGW)
YWBFC = VPAQ W*VPAQ W*SIN(ALPHAG) / CSA*CSA*CSA
İŞE = (ISQPL .AND. JBEW) = İİFEQ (JBEW .AND. JBEW) .OR. DBFC / YWBFC
790 CONTINUE

C
C PLOTTING ROUTINE FOR THE SINGLE PARTICLE TRAJECTORIES.
C
905 IF (PTL .LT. 1.0) GO TO 52
111 L1 = 1 - 1
112 IMCH = (L1 - 1) / IGSRP + 1
113 DO 93 IA = 1, IMCH
114 YANG = LALPHA(A) * EIR
115 ALPHA(A) = LALPHA(A) * EIR
116 ANP = (YANG + RANGE) * SF + 1.0
117 NMPH = (PHASE(A) + RANGYP) * SPPH + 1.0
118 IF (NP .LT. 1.0 OR. NNP .GT. 79) GO TO 907
119 GR(NMPH) = YMULT
907 IF (NPFFH .LT. 1.0 OR. NPFFH .GT. 79) GO TO 202
202 WRITE(6, 203) YANG, VIIFF (IA), PHASE (IA), YLT (IA), TIM (IA), GR, VPARA (IA),
11ALPHA (IA)
203 FORMAT (F7.1, F6.2, F9.1, F6.2, F6.2, F7.0, F5.2)
205 IF (NP .LT. 1.0 OR. NNP .GT. 79) GO TO 908
206 GR(NPFFH) = BLANK
908 IF (NPFFH .LT. 1.0 OR. NPFFH .GT. 79) GO TO 54
209 GR(NPFFH) = BLANK
300 94 GR(NF) = VERLN
301 93 CONTINUE
302 52 IF (WHAT) GO TO 702
303 702 CONTINUE
304 701 CONTINUE
C
C PAINTING OF THE DISTRIBUTION FUNCTION.
C
314 IF (ISQPL .LT. 1.0) GO TO 524
315 WRITE (6, 604) (PICHU(KU), KU = 1, 32, 2)
316 604 FORMAT (1T, 'EQUATORIAL DISTRIBUTION FUNCTION' / VPARALEL(KH/SEC)
1'T, 50X, 'PITCH ANGLE (DEG)' / '7X, 15F6.1/9X, 32 ('1')
317 DO 601 INF = 1, NFP
318 IF (IBEQ .EQ. NVBP) GO TO 524
319 VILACT = VILQV (IBEQ)/1000.0
320 DO 605 JBEQ = 1, 33
321 605 IF (JBEQ .EQ. NVBP) GO TO 524
322 603 FORMAT (1X, F6.0, '---', 133 (12, 1))
323 WRITE (6, 603) VILACT, (IBEQ, JBEQ .EQ. 1, 33)
324 601 CONTINUE
325 524 CONTINUE
326 STOP
327 END
FIGURE D.1 Typical program output for the DATA input shown on the top.

( B = 10 my, \( \phi = 67.75^\circ \), \( \phi = 90^\circ \))

a) Particle trajectory.

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(b) The full distribution. The distribution in this case is $f(v, \alpha) = 1$ above the losscone and is zero below the losscone. The one particle for which the trajectory is shown in the previous page has moved from $\alpha_{eq} = 6.75^\circ$ and $v_{eq} = 18992$ km/sec to $\alpha_{eq} = 2.75^\circ$ and $v_{eq} = 19041$ km/sec. The value of the distribution at the new location is modified in accordance with the description given on page 121.
REFERENCES


Angerami, J. J., Whistler duct properties deduced from VLF observations made with the OGO-3 satellite near the magnetic equator, *J. Geophys. Res.*, 75, 6115, 1970.


Appleton, E. V., URSI Reports, Washington, 1927.


Buneman, O., Class notes for Introduction to Plasma Physics, Stanford University, 1973.


Crystal, T. L., Nonlinear currents stimulated by monochromatic whistler mode (WM) waves in the magnetosphere, Tech. Rept. No. 3465-4, Radioscience Laboratory, Stanford Electronics Labs., Stanford University, Stanford, Calif., 1975.
Edgar, B. C., The structure of the magnetosphere as deduced from magnetospherically reflected whistlers, Tech. Rept. No. 3438-2, Stanford Electronics Labs., Radioscience Laboratory, Stanford University,
Stanford, Calif., 1972.


Helliwell, R. A., J. P. Katsufrakis, T. F. Bell and R. Raghuram, VLF line radiation in the earth's magnetosphere and its association with


Lyons, L. R., Pitch angle and energy diffusion coefficients from resonant interactions with ion-cyclotron and whistler waves, *J. Plasma Phys.*, 12, 417, 1974b.


Matsumoto, H., Theoretical studies on whistler mode wave particle interactions in the magnetospheric plasma, Kyoto Univ., Kyoto, Japan, 1972.


