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A fast method to determine the nose frequency and minimum group delay of a whistler when the causative spheric is unknown

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Abstract—Using Bernard's (1973) approximation of the ducted whistlers, a fast and simple method is developed for the determination of whistler nose frequency (f_n) and travel time at the nose (t_n) when both the nose frequency and the causative spheric are not directly identifiable on whistler spectrograms. This method requires measurements of time delays at only three frequencies with respect to an arbitrary time origin, and the formulas for f_n and t_n involve only simple algebraic functions. The error for f_n is comparable to the error made in measuring f_n visually in the case of nose whistlers, and a value of $\Delta t_n/t_n$ of order 5 per cent or less is achievable.

1. Introduction

The ground-based whistler technique has provided a powerful tool for studies of the structure and dynamics of the inner magnetosphere (e.g. Angerami and Carpenter, 1966; Park and Carpenter, 1970; Carpenter *et al.*, 1972; Park, 1970, 1972b). This technique is based on the measurements of the nose frequency f_n and the corresponding minimum time delay t_n. However, the observational whistler data frequently do not exhibit the portion of the whistler wave near the nose frequency or the causative spheric or both, and in such cases curve fitting methods are needed to estimate the values of f_n and t_n.

In the methods described by Smith and Carpenter (1961), Brice (1965), Dowden and Allcock (1971) and Bernard (1973), the causative spheric must first be identified. This is not necessary in the methods of Sagredo *et al.* (1973) and Rycroft and Mathur (1973), which are based on the Dowden and Allcock (1971) approximation of the whistler dispersion $D \equiv t\sqrt{f}$ ($1/D$ is a linear function of frequency in this approximation). In the method of Sagredo *et al.*, the causative spheric can be estimated if f_n is known. Rycroft and Mathur's method, on the other hand, does not require a knowledge of f_n. Both methods are satisfactory under certain conditions, but the main disadvantage is that they require a large number of scaled data points.

In this paper, Bernard's (1973) approximation is used to calculate f_n and t_n from three points along the whistler trace. Knowledge of the causative spheric and the nose is not required. This method allows simple and speedy data processing with high precision.

The formulas for f_n and t_n are developed in the next section, followed by discussion of their applications and limitations in Section 3.

2. Nose Extension Method

Bernard (1973) introduced an approximate dispersion function for ducted whistlers:

$$D \equiv t\sqrt{f} \sim D_0 \frac{f_{HE} - Af}{f_{HE} - f} \tag{1}$$

where

D_o: zero-frequency dispersion
f_{HE}: equatorial electron gyrofrequency

$$A = \frac{3\Lambda_n - 1}{\Lambda_n(1 + \Lambda_n)}$$

$$\Lambda_n = \frac{f_n}{f_{HE}}.$$

At the nose frequency and with this approximation:

$$D_n = t_n\sqrt{f_n} = D_o \frac{2}{1 + \Lambda_n}. \tag{2}$$

For a typical diffusive equilibrium model ($T_e = T_i = 1600°K$; 90% O^+, 8% H^+ and 2% He^+ at 1000 km altitude) and a dipole field model $\Lambda_n \cong 0{\cdot}370$ and $A \cong 0{\cdot}217$ (Park, 1972a; Bernard, 1973). The effect of propagation through conjugate ionospheres can be taken into account either by Park's (1972a) method or by using modified values of Λ_n and A. For 'average' ionospheric conditions the values $\Lambda_n = 0{\cdot}377$ and $A = 0{\cdot}252$ may be used as first order corrections (Bernard, 1973).

(a) *Determination of f_n*

The nose frequency can be extrapolated from measurements of frequency f and time t with respect to an arbitrary time origin at three points on the whistler trace. They are subscripted 1, 2 and 3.

$$(t_1 - t_2)t_3 + (t_2 - t_3)t_1 + (t_3 - t_1)t_2 = 0. \tag{3}$$

Substituting equation (1) into equation (3), we obtain

$$E_{123}\frac{f_{HE} - Af_3}{f_{HE} - f_3} + E_{231}\frac{f_{HE} - Af_1}{f_{HE} - f_1} + E_{312}\frac{f_{HE} - Af_2}{f_{HE} - f_2} = 0 \tag{4}$$

where

$$E_{ijk} = \frac{t_i - t_j}{\sqrt{f_k}} = \frac{\Delta t_{ij}}{\sqrt{f_k}}.$$

Equation (4) can be rearranged to read

$$f_{HE}^3 + pf_{HE}^2 + qf_{HE} + r = 0 \tag{5}$$

where

$$p = -\sum f_i + (1 - A)\frac{\sum E_{ijk}f_k}{\sum E_{ijk}}$$

$$q = A\sum f_i f_j + (1 - A)\frac{\sum E_{ijk}f_i f_j}{\sum E_{ijk}} \tag{6}$$

and

$$r = -Af_1 f_2 f_3.$$

$\sum$ represents the sum of all possible combinations of three subscripted variables. For example,

$$\sum f_i f_j = f_1 f_2 + f_2 f_3 + f_3 f_1$$

and

$$\sum E_{ijk} = E_{123} + E_{231} + E_{312}.$$

Equation (5) has three solutions; however, only one solution meets the physical requirement that $f_{HE} \geqslant 2f_i$. It is a necessary condition for the trapping of whistlers in ducts of enhanced ionization (SMITH, 1961) and is also confirmed by ground-based observations (CARPENTER, 1968).

Equation (5) can be solved analytically by standard procedures (see, for example, SELBY, 1968). An alternative is to simplify the calculations by the following procedure.

Since $f_{HE}^3 \gg Af_1f_2f_3$ we write:

$$f_{HE}^2 + pf_{HE} + q \sim 0. \tag{7}$$

Approximate solutions for f_{HE} are

$$f_{HE} \sim f_a = -\tfrac{1}{2}[p \pm (p^2 - 4q)^{1/2}]. \tag{8}$$

Again we choose the solution which meets the requirement that $f_{HE} \geqslant 2f_i$. This solution is then improved upon by letting

$$f_{HE} = f_a + \varepsilon \tag{9}$$

where

$$\varepsilon \ll f_{HE}.$$

We substitute equation (9) into equation (5) and then simplify by using equation (7) and ignoring terms involving ε^2 and ε^3. As a result, we obtain

$$\varepsilon = -\frac{r}{3f_a^2 + 2pf_a + q} = \frac{-r}{2f_a^2 + pf_a}. \tag{10}$$

The nose frequency is obtained from the relationship

$$f_n = \Lambda_n f_{HE}. \tag{11}$$

(b) *Determination of* t_n

Once f_n is determined, t_n can be calculated from any combination of two points on the whistler trace, provided their time delays are different.

From equations (2) and (1), we can write

$$t_n = \frac{D_o}{\sqrt{f_n}} \frac{2}{1 + \Lambda_n} \tag{12}$$

and

$$t_i - t_j = D_o \left[\frac{1}{\sqrt{f_i}} \frac{f_{HE} - Af_i}{f_{HE} - f_i} - \frac{1}{\sqrt{f_j}} \frac{f_{HE} - Af_j}{f_{HE} - f_j} \right]. \tag{13}$$

Combining equations (12) and (13) and rearranging, we obtain

$$t_n = \frac{2}{1+\Lambda_n}\frac{\Delta t_{ij}}{\sqrt{f_n}}\left\{\frac{1}{\sqrt{f_i}}\frac{f_{HE}-Af_i}{f_{HE}-f_i} - \frac{1}{\sqrt{f_j}}\frac{f_{HE}-Af_j}{f_{HE}-f_j}\right\}^{-1}. \tag{14}$$

If one of the two points is at the nose, equation (14) can be simplified

$$t_n = \Delta t_{nj}\varphi(F) \tag{15}$$

where

$$\varphi(F) = \left\{1 - \frac{1}{2\sqrt{F}}\frac{1+\Lambda_n-(3\Lambda_n-1)F}{1-\Lambda_n F}\right\}^{-1}$$

and

$$F = \frac{f_j}{f_n}.$$

3. Applications and Limitations

The above method was tested by applying it to six very well defined nose whistlers recorded at Eights, Antarctica (75°S 77°W geographic; $L = \sim 3{\cdot}7$) on 26 March, 10 June and 25 June 1965. Due to the repeatability of these whistlers over a range of time, a great accuracy in the values of f_n measured visually was possible. The causative spheric was clearly identifiable in all cases, and nose frequencies ranged from ~4 to 16 kHz. For each whistler, time delays were measured with respect to an arbitrary origin (not the causative spheric) at many frequencies. Various combinations of three measured points were then used to calculate f_n and t_n. The results show that the errors in this method depend upon the choice of measured points. In general, large errors must be expected if the measurements are made at low frequencies where dispersion is nearly constant with frequency. The errors also depend in complex ways upon frequency and time separation between scaled points. We introduce below two parameters which can be used as indicators of probable errors in f_n and t_n.

We can see from equations (5) and (6) that a small measurement error will give large errors in f_n if $\sum E_{ijk}$ approaches zero, or correspondingly if

$$\frac{\Delta t_{12}}{\sqrt{f_3}} + \frac{\Delta t_{23}}{\sqrt{f_1}} + \frac{\Delta t_{31}}{\sqrt{f_2}} \to 0. \tag{16}$$

By introducing a parameter S and using $\Delta t_{23} = -(\Delta t_{31} + \Delta t_{12})$, equation (16) can be rewritten:

$$S \equiv \frac{\Delta t_{21}(1-\sqrt{f_1/f_3})}{\Delta t_{31}(1-\sqrt{f_1/f_2})} \to 1 \tag{17}$$

provided $t_1 \neq t_3$.

Figure 1 shows how the error in f_n depends on S. As expected, the error increases as S approaches unity. $S = 0$ when two out of three points have the same time delays. The error is impressively low for $S \leq 0$, that is, when a portion of the whistler above the nose can be identified. The error for $S > 0$ in this figure is plotted

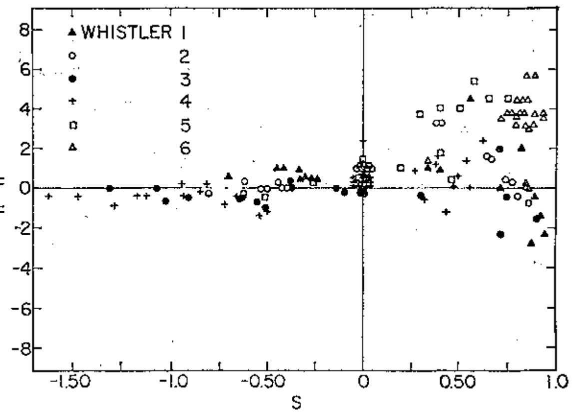

Fig. 1. Percentage error in extrapolated values of f_n plotted against the parameter $S = \Delta t_{21}(1 - \sqrt{f_1/f_3})[\Delta t_{31}(1 - \sqrt{f_1/f_2})]^{-1}$ for six actual nose whistlers.
Whistler No. 1: $t_n = 2{\cdot}63$ s $\pm$ 1·5%; $f_n = 4{\cdot}25$ kHz $\pm$ 1%.
$t_n = 2{\cdot}49$ s $\pm$ 1%; $f_n = 4{\cdot}41$ kHz $\pm$ 1%.
$t_n = 2{\cdot}34$ s $\pm$ 1·5%; $f_n = 4{\cdot}55$ kHz $\pm$ 2%.
$t_n = 0{\cdot}97$ s $\pm$ 1·5%; $f_n = 9{\cdot}64$ kHz $\pm$ 3%.
$t_n = 0{\cdot}69$ s $\pm$ 1·5%; $f_n = 12{\cdot}96$ kHz $\pm$ 2%.
$t_n = 0{\cdot}62$ s $\pm$ 1·2%; $f_n = 15{\cdot}26$ kHz $\pm$ 3·5%.
Whistler No. 6 shows larger errors than other whistlers because S is positive and approaching unity.

only with combinations of points below the nose to show the application of this method in the case of whistlers whose portions above the noses are not identifiable.

By similar reasoning, we expect large errors in t_n when the quantity inside the bracket in equation (14) approaches zero. We introduce a parameter

$$R_t \equiv \frac{\dfrac{1}{\sqrt{f_i}}\dfrac{f_{HE} - Af_i}{f_{HE} - f_i}}{\dfrac{1}{\sqrt{f_j}}\dfrac{f_{HE} - Af_j}{f_{HE} - f_j}} = \frac{t_i}{t_j} \tag{18}$$

and when R_t approaches unity, large errors should be expected.

Figure 2 shows the percentage error in t_n plotted against R_t. Again, as predicted, the error increases when R approaches unity. More than 85 per cent of the combinations show error less than ± 5 per cent. Errors in f_n and t_n are not mutually independent. This is illustrated with a scatter plot of $(\Delta t_n/t_n)$ vs. $(\Delta f_n/f_n)$ in Fig. 3, where a small negative correlation is apparent.

With f_n and t_n, magnetospheric parameters such as equatorial radius of the whistler duct (L), equatorial electron concentration (n_e), and tube content (N_T) can be calculated by Park's (1972) methods. Errors in these magnetospheric parameters due to the nose extension method may be estimated as follows:

$$\frac{\Delta L}{L} = -\frac{1}{3}\frac{\Delta f_n}{f_n}$$

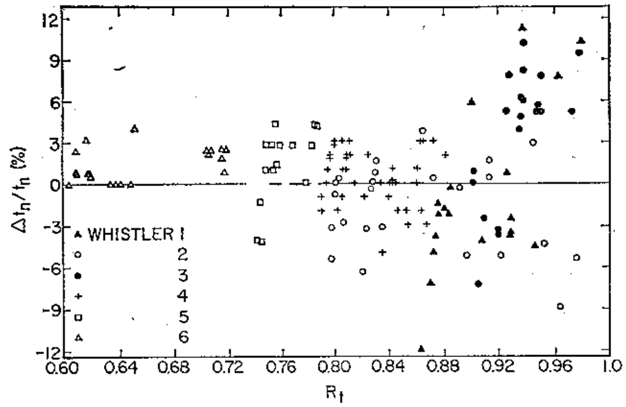

Fig. 2. Percentage error in extrapolated values of t_n plotted against the parameter $R_t = t_i/t_j$ for the six whistlers of Fig. 1.

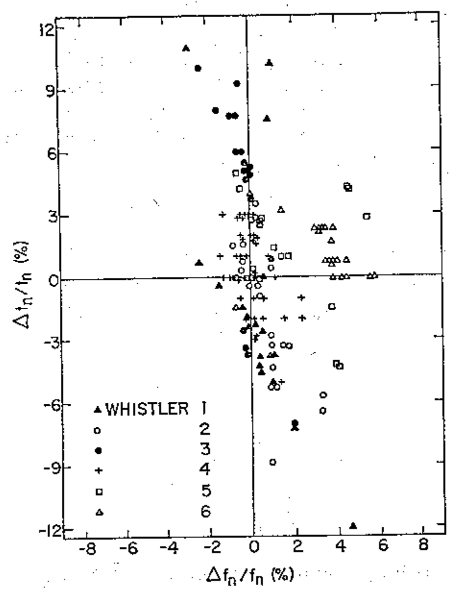

Fig. 3. A plot of percentage error in t_n vs. percentage error in f_n. There is a negative correlation between the two errors.

$$\frac{\Delta n_e}{n_e} = \frac{8}{3}\frac{\Delta f_n}{f_n} + 2\frac{\Delta t_n}{t_n}$$

$$\frac{\Delta N_T}{N_T} = 2\frac{\Delta t_n}{t_n} + \frac{4}{3}\frac{\Delta f_n}{f_n}.$$

If a 2 per cent error of f_n and a generous 5 per cent error of t_n are obtained by this method, and if as pointed out before, the errors of f_n and t_n are of opposite sign (Fig. 3), then

$$\frac{\Delta L}{L} = +0{\cdot}67\%$$

$$\frac{\Delta n_e}{n_e} = \frac{-16}{3}\% + 10\% = 4{\cdot}67\%$$

$$\frac{\Delta N_T}{N_T} = -\frac{8}{3}\% + 10\% = 7{\cdot}4\%.$$

These errors are comparable to errors from other sources (PARK, 1972a).

In other situations, the tube content is not of direct interest and only f_n is needed. This is true in whistler direction findings (COUSINS, 1972) or in certain measurements of magnetospheric east–west convection electric fields using cross-L drifts of whistler ducts (CARPENTER *et al.*, 1972). In these cases the error in f_n deduced by this method (see Fig. 1) is comparable to the expected error in f_n from visual measurements (BERNARD, 1973).

As discussed above, R_t and S should be far away from unity to have accurate t_n and f_n. Because of the R_t criterion, two points should be chosen far apart in time. If only the portion below the nose is identifiable, the third point should be taken approximately in the middle ($\Delta t_{12} \simeq \Delta t_{23}$) to have the smallest S and therefore the best f_n. If the portion of the whistler just above the nose can be identified as well as the portion of the whistler below the nose, the combination should be chosen such that $S \leq 0$ (for example, $f_1 > f_2 > f_3$ and $t_2 \leq t_1 < t_3$). This is particularly true if the emphasis is on f_n. Otherwise, the choice discussed earlier always gives very good values of t_n and f_n.

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